On the Computational Complexity of Minimal Cumulative Cost Graph Pebbling

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Structure of Talk

- Background
- Graph Pebbling
- “Graph Reducibility”
- Open Problems

Background

- ❖ Users tend to pick weak passwords

TOP 20
MOST COMMON PASSWORDS
(as a percentage of all passwords)

1.	123456	4.1%	11.	login	0.2%
2.	password	1.3%	12.	welcome	0.2%
3.	12345	0.8%	13.	loveme	0.2%
4.	1234	0.6%	14.	hottie	0.2%
5.	football	0.3%	15.	abc123	0.2%
6.	qwerty	0.3%	16.	121212	0.2%
7.	1234567890	0.3%	17.	123654789	0.2%
8.	1234567	0.3%	18.	flower	0.2%
9.	princess	0.3%	19.	passw0rd	0.2%
10.	solo	0.2%	20.	dragon	0.1%

Source: SkyHigh

Background

❖ Users tend to pick weak passwords

❖ Server attacks are inevitable

Entity	Year	Records	Organization type	Method
Accendo Insurance Co.	2011	175,350	healthcare	poor security
Adobe Systems	2014	152,000,000	tech	hacked
Advocate Medical Group	2013	4,000,000	healthcare	lost / stolen media
Affinity Health Plan, Inc.	2009	344,579	healthcare	lost / stolen media
Ameritrade	2005	200,000	financial	lost / stolen media
Ankle & Foot Center of Tampa Bay, Inc.	2010	156,000	healthcare	hacked
Anthem Inc.	2015	80,000,000	healthcare	hacked
AOL	2004	92,000,000	web	inside job, hacked
AOL	2006	20,000,000	web	accidentally published
AOL	2014	2,400,000	web	hacked
Apple, Inc./BlueToad	2012	12,367,232	tech, retail	accidentally published
Apple	2013	275,000	tech	hacked
Apple Health Medicaid	2016	91,000	healthcare	poor security
Ashley Madison	2015	32,000,000	web	hacked
AT&T	2008	113,000	telecoms	lost / stolen computer
AT&T	2010	114,000	telecoms	hacked
Auction.co.kr	2008	18,000,000	web	hacked
Australian Immigration Department	2015	G20 world leaders	government	accidentally published
Automatic Data Processing	2005	125,000	financial	poor security

Entity	Year	Records
Yahoo	2013	1,000,000,000
Yahoo	2014	500,000,000
Friend Finder Networks	2016	412,214,295
Massive American business hack including 7-Eleven and Nasdaq	2012	160,000,000
Adobe Systems	2014	152,000,000
eBay	2014	145,000,000
Heartland	2009	130,000,000
Rambler.ru	2012	98,167,935
TK / TJ Maxx	2007	94,000,000
AOL	2004	92,000,000
Anthem Inc.	2015	80,000,000
Sony PlayStation Network	2011	77,000,000
JP Morgan Chase	2014	76,000,000
National Archives and Records Administration (U.S. military veterans' records)	2009	76,000,000
Target Corporation	2014	70,000,000
Home Depot	2014	56,000,000

YAHOO!

OHIO STATE

Gmail by Google

Adobe

AMERICA Online

ASHLEY MADISON.COM

BLIZZARD ENTERTAINMENT

eHarmony

LastPass ****

ebay

Dropbox

LinkedIn

PlayStation

UBER

U.S. DEPARTMENT OF HOMELAND SECURITY

DNC
DEMOCRATIC
NATIONAL
COMMITTEE

COX
COMMUNICATIONS

Dropbox

PlayStation

User	Password
Stephen	auhsoJ
Lisa	hsifdrowS
James	1010NO1Z
Harry	sinocarD tupaC
Sarah	auhsoJ

User	Password Hash
Stephen	39e717cd3f5c4be78d97090c69f4e655
Lisa	f567c40623df407ba980bfad6dff5982
James	711f1f88006a48859616c3a5cbcc0377
Harry	fb74376102a049b9a7c5529784763c53
Sarah	39e717cd3f5c4be78d97090c69f4e655

User	Random Salt	Password Hash
Stephen	06917d7ed65c466fa180a6fb62313ab9	b65578786e544b6da70c3a9856cdb750
Lisa	51f2e43105164729bb46e7f20091adf8	2964e639aa7d457c8ec0358756cbffd9
James	fea659115b7541479c1f956a59f7ad2f	dd9e4cd20f134dda87f6ac771c48616f
Harry	30ebf72072134f1bb40faa8949db6e85	204767673a8d4fa9a7542ebc3eceb3a2
Sarah	711f51082ea84d949f6e3efecf29f270	e3afb27d59a34782b6b4baa0c37e2958

Background

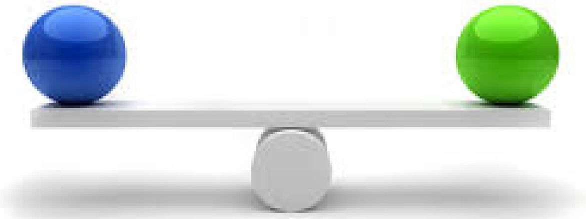
- Users tend to pick weak passwords
- Server attacks are inevitable
- Try to mitigate offline attacks
- Specialized hardware (ASIC) can compute 10^{12} hashes per second.

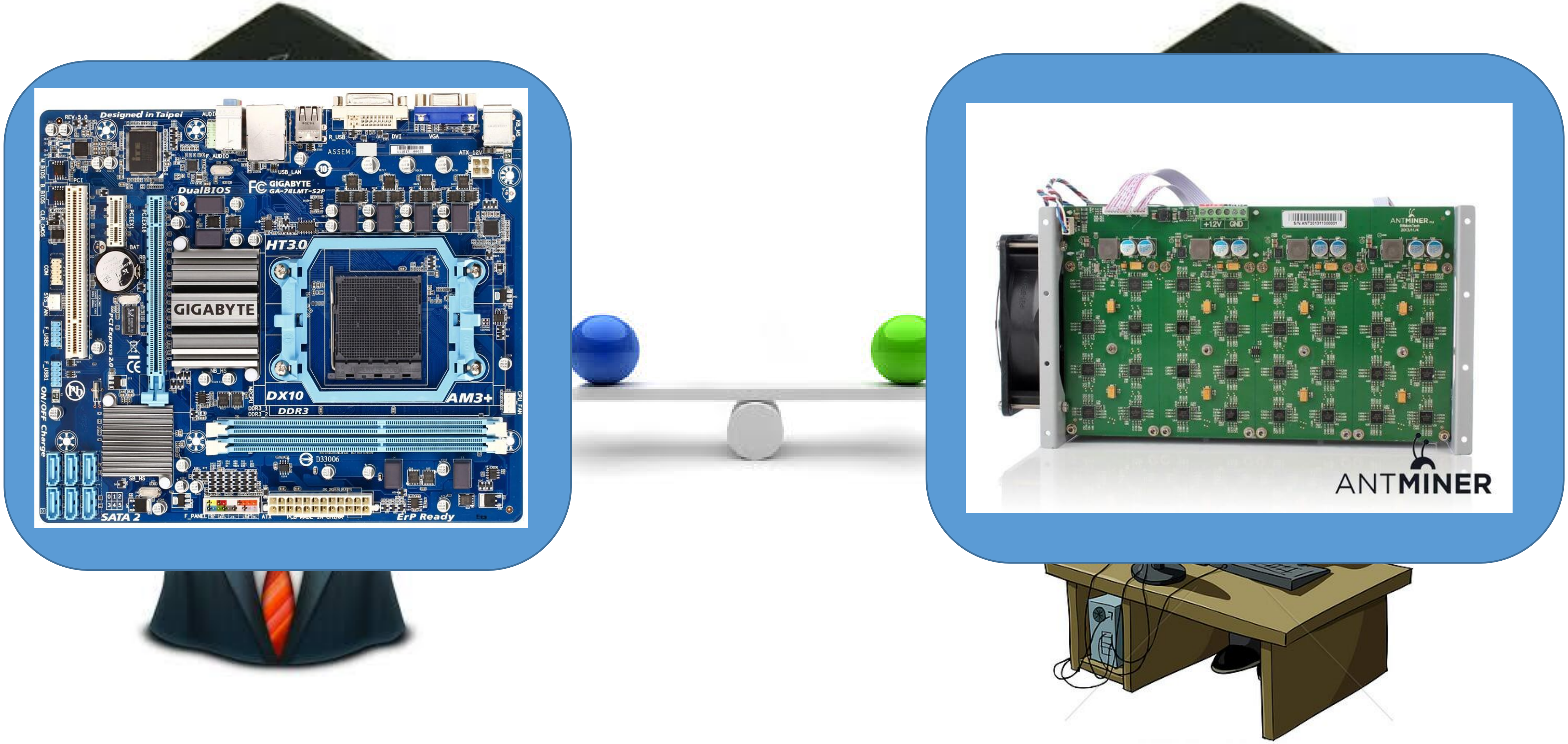
Hash Function Goals

- “Moderately Expensive” to compute
- Expensive to compute on ASIC
- Fast and cheap on PC


AT27C010
70J1
0307

AT27C010
70J1
0307

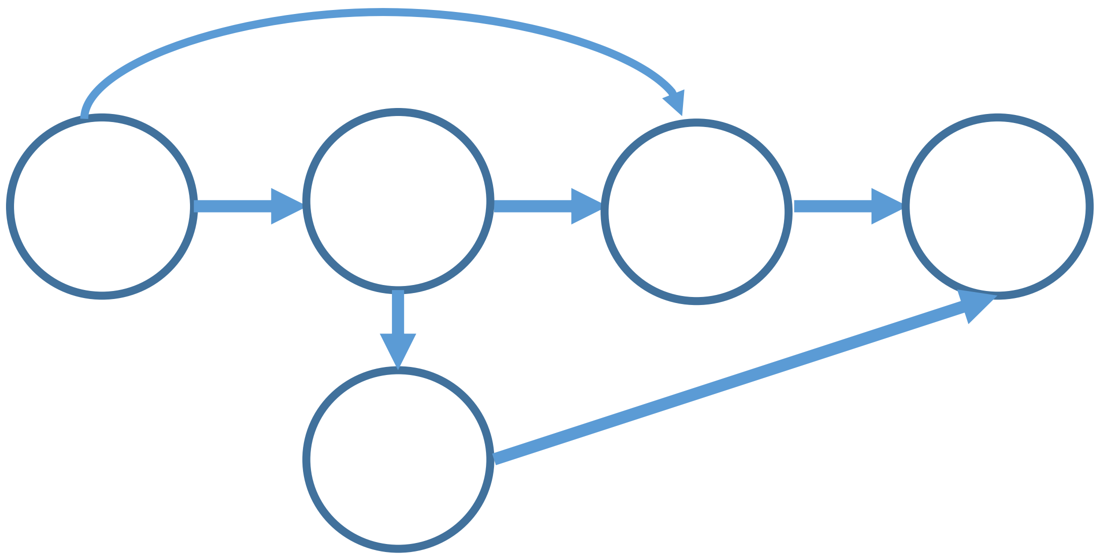




GIGABYTE
ANTMINER

iMHFs

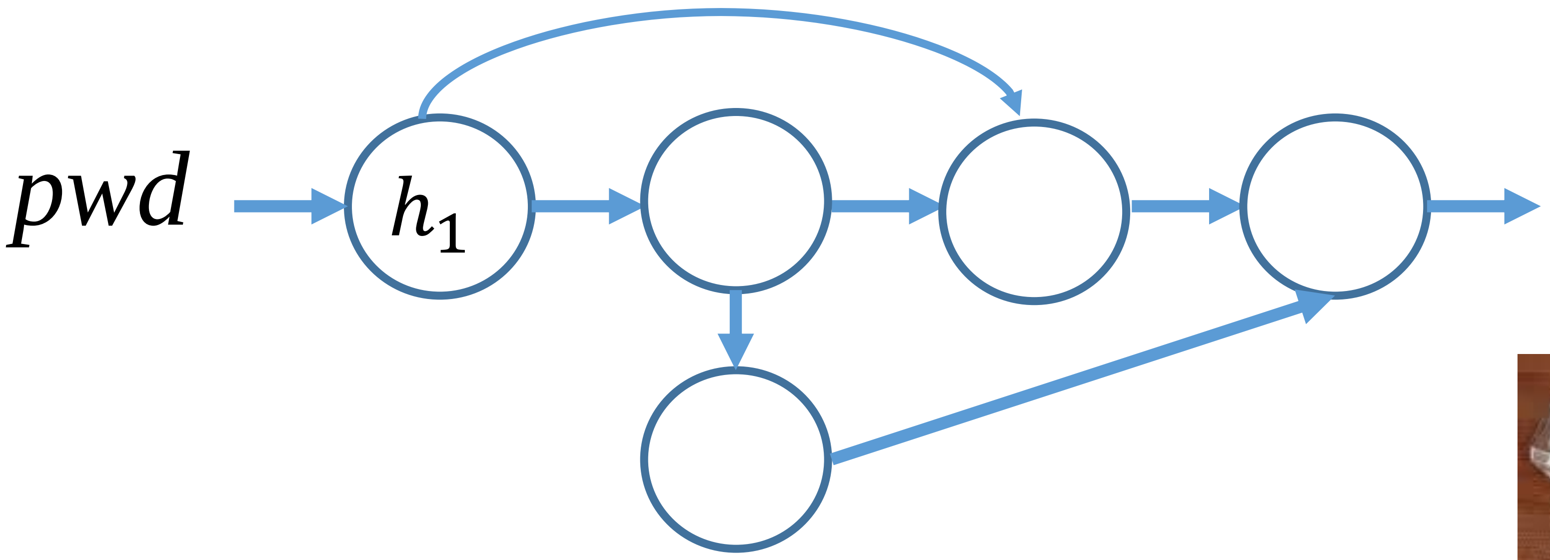
- Data-independent memory hard functions require comparatively more resources for adversaries to compute

iMHFs



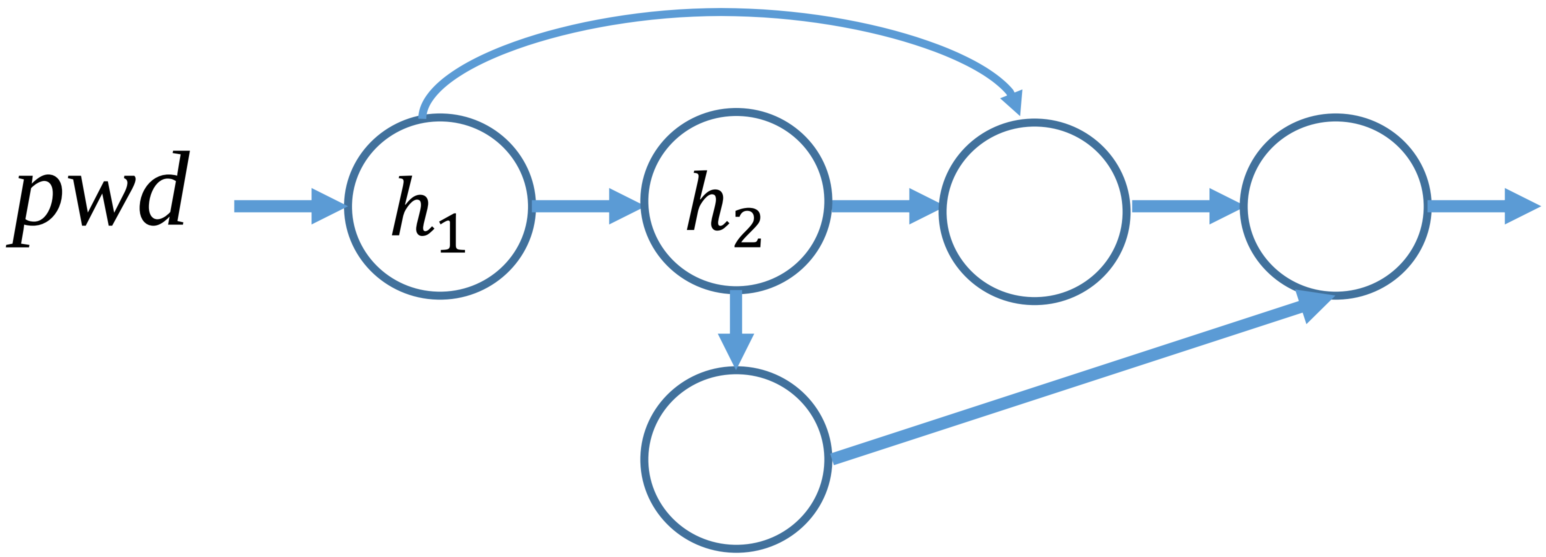
Hash function: H

iMHFs



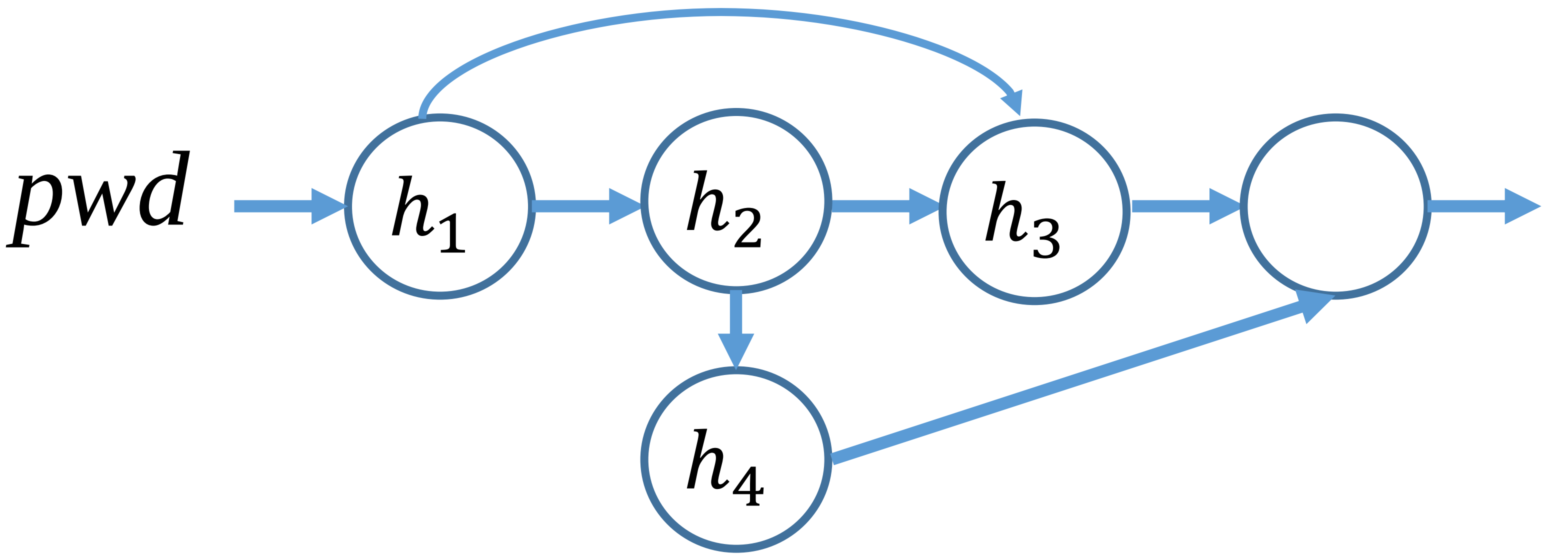
$$h_1 = H(pwd, salt)$$

iMHFs



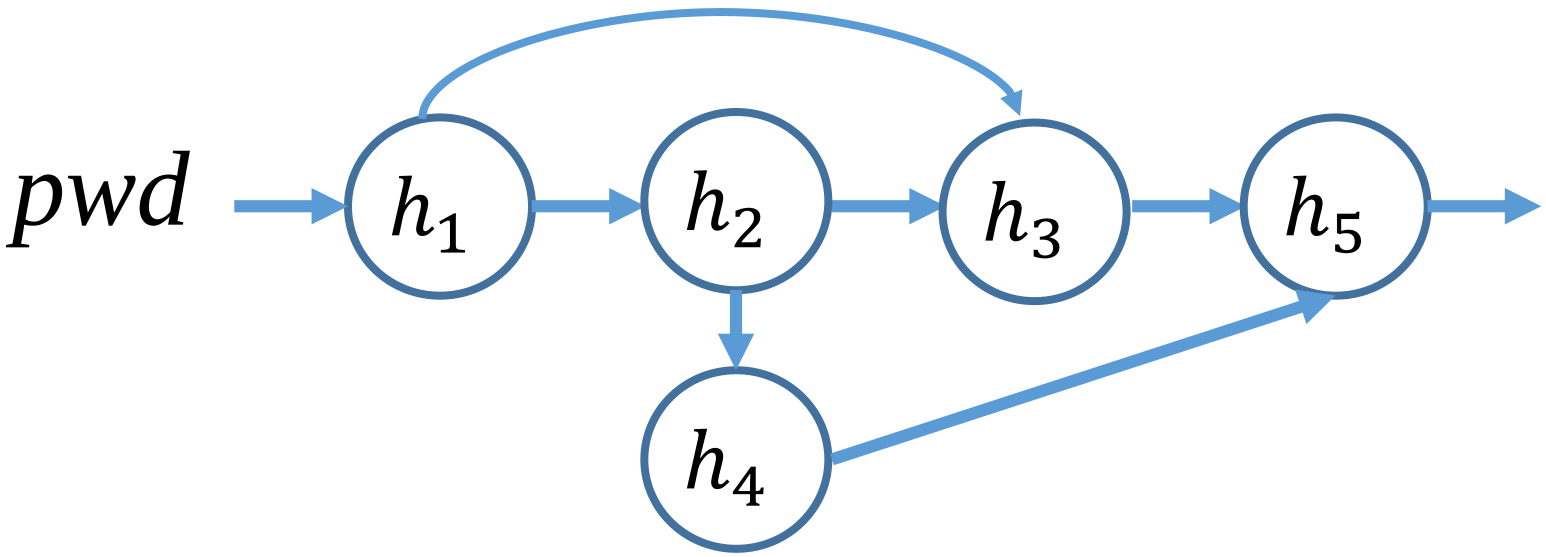
$$h_2 = H(h_1)$$

iMHFs



$$h_3 = H(h_1, h_2), \qquad h_4 = H(h_2)$$

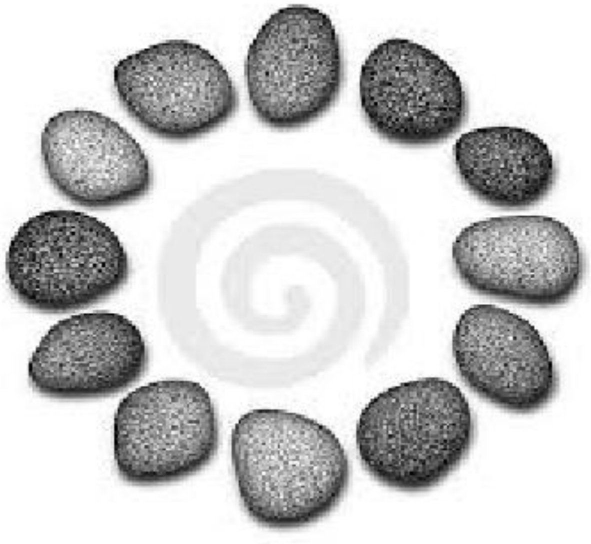
iMHFs



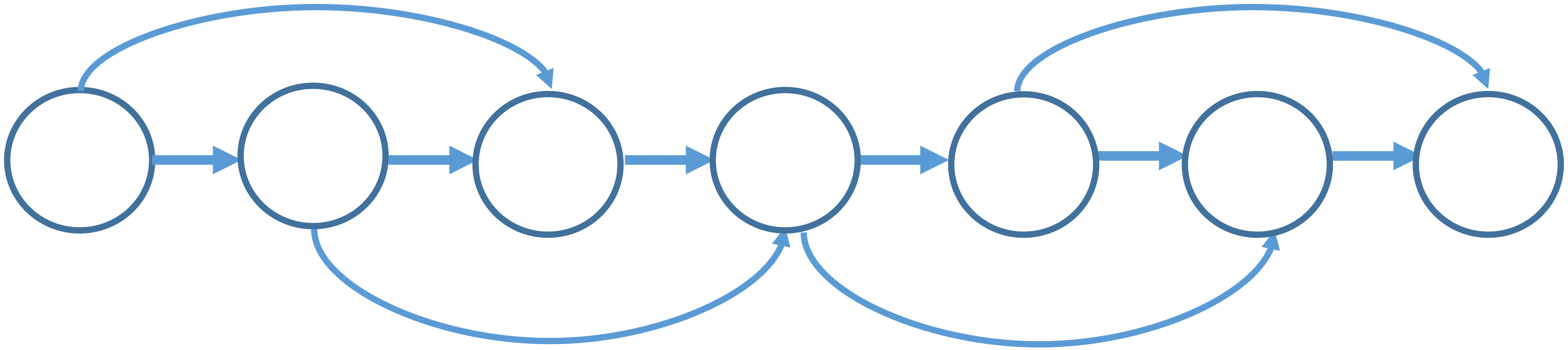
$$h_5 = H(h_3, h_4)$$

iMHFs

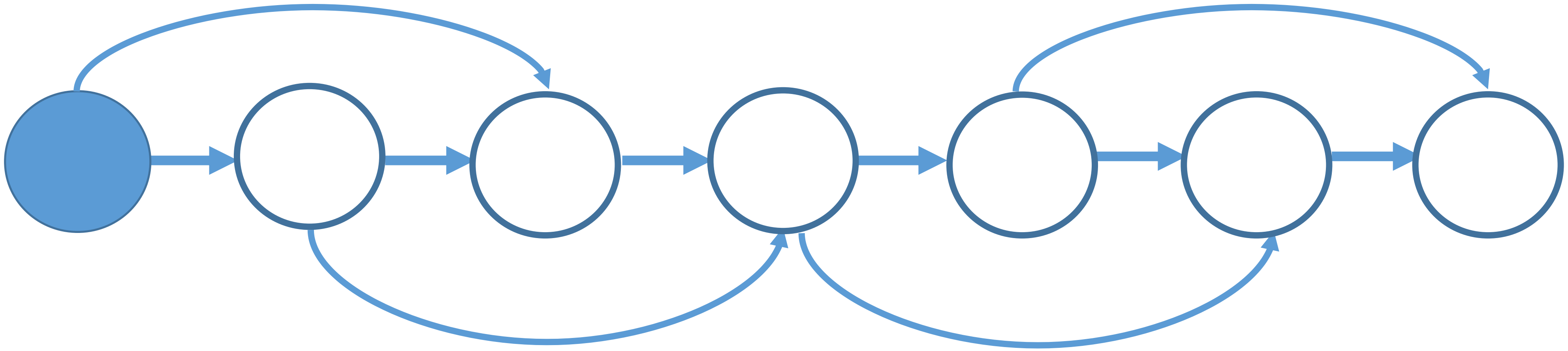
- Data-independent memory hard functions require comparatively more resources for adversaries to compute
- Calculating an iMHF can be modeled as graph pebbling



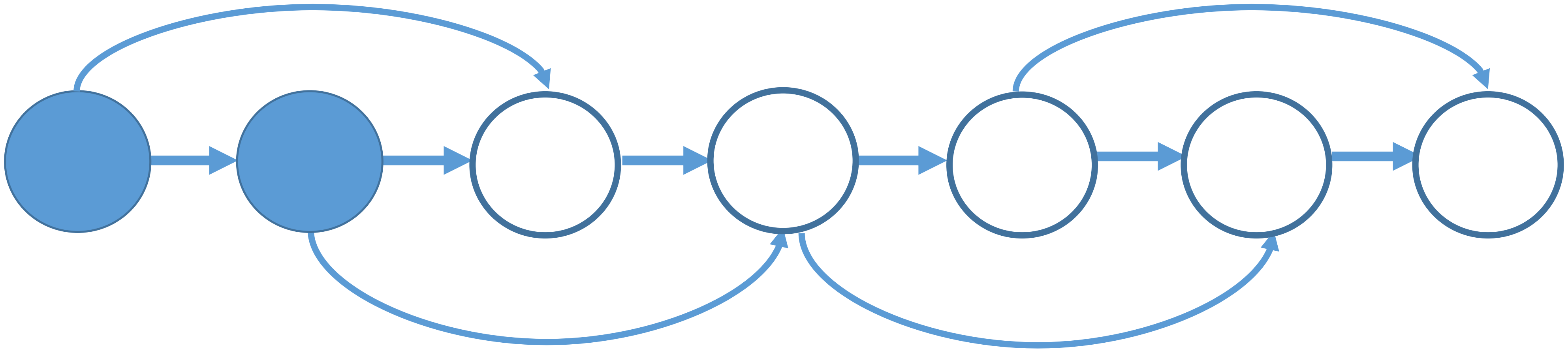
Graph Pebbling



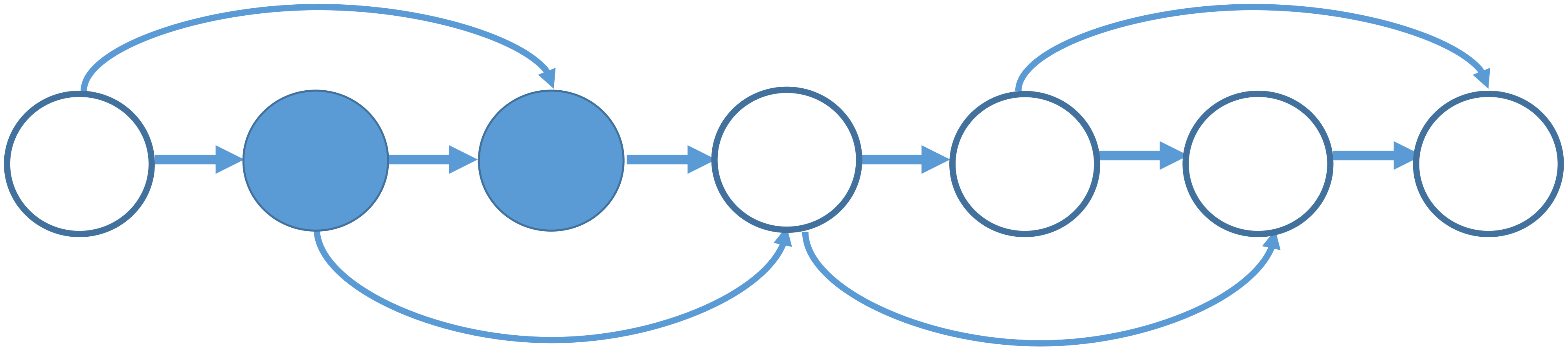
Graph Pebbling



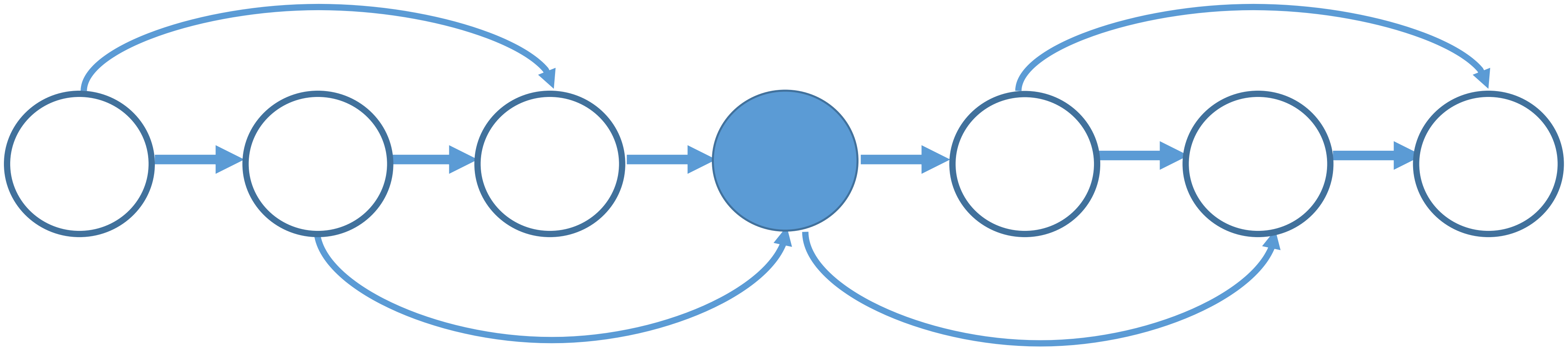
Graph Pebbling



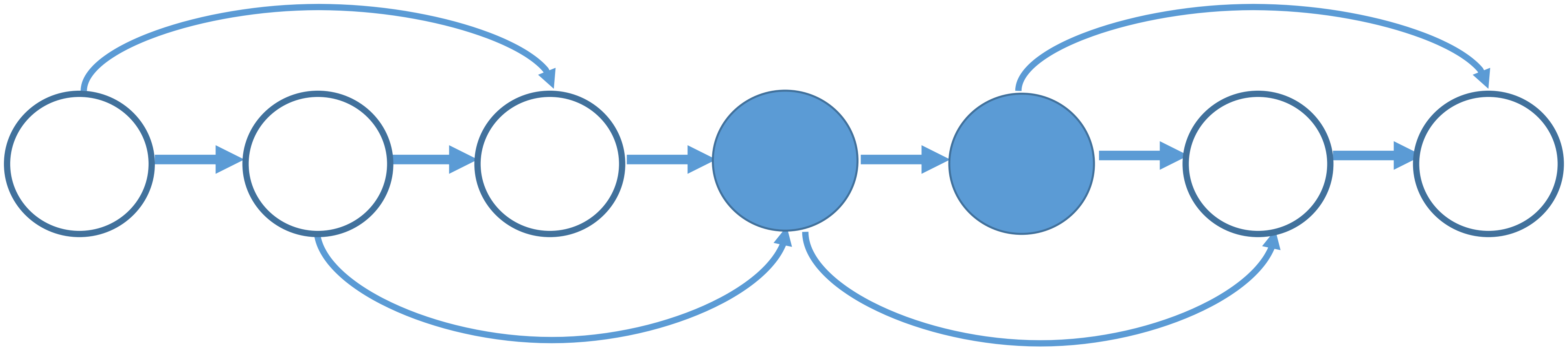
Graph Pebbling



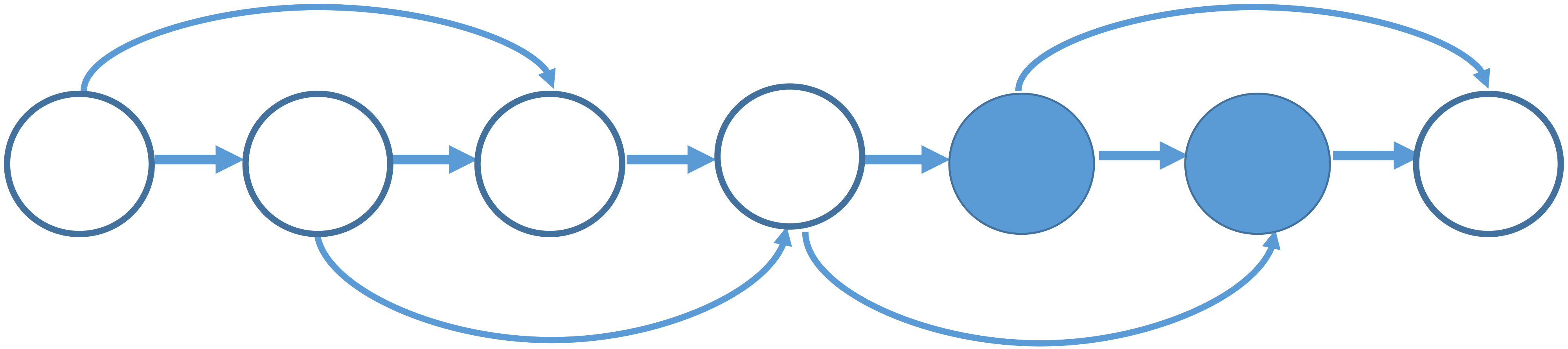
Graph Pebbling



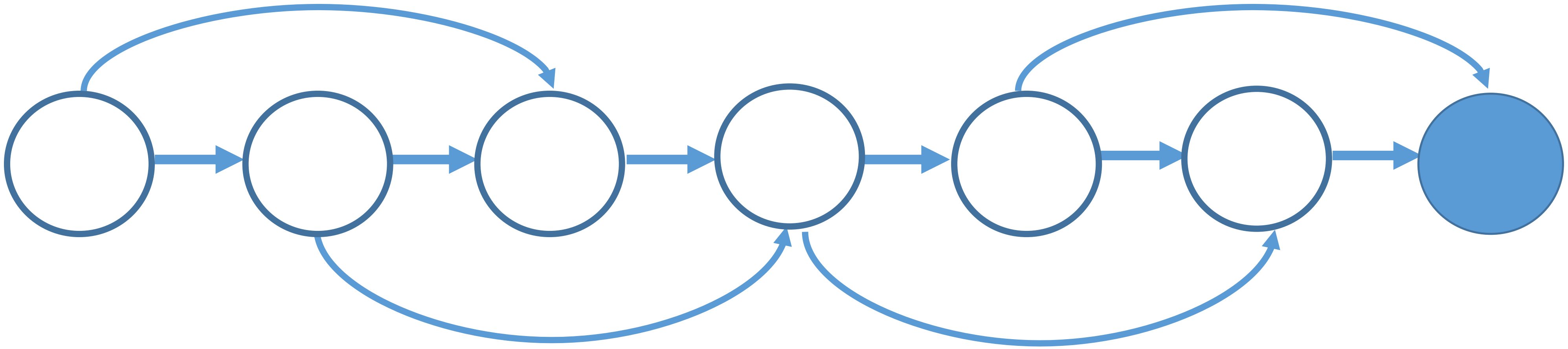
Graph Pebbling



Graph Pebbling

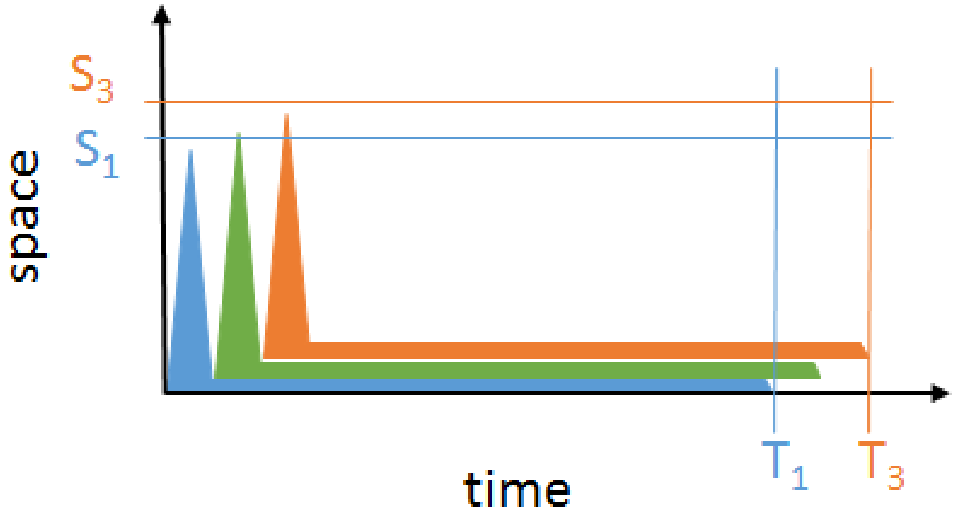


Graph Pebbling



iMHFs

- How to evaluate iMHF?
- ST-complexity: number of pebbles × number of steps
  - 2 pebbles × 7 steps = 14
- ST-complexity can scale badly with multiple evaluations [AS15]



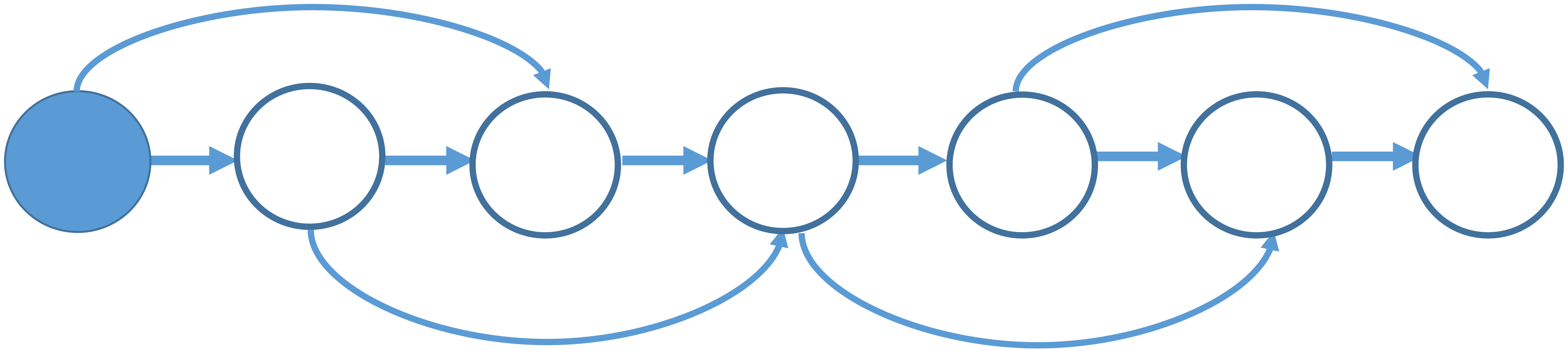
iMHFs

❖ [AS15] ∃ function f such that: $ST(\sqrt{n}\ instances\ of\ f) = O(ST(f))$

“What’s an anagram of Banach-Tarski?”

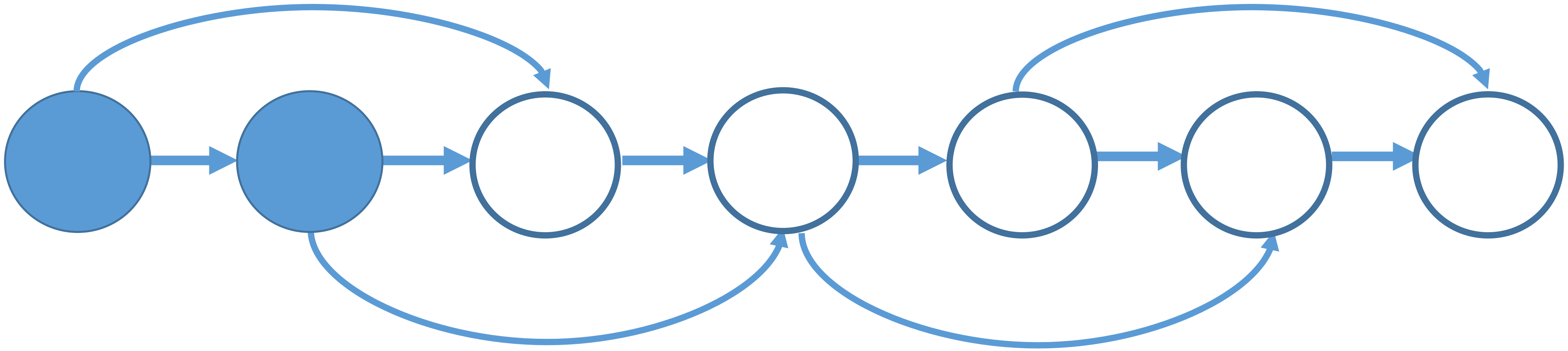
“Banach-Tarski Banach-Tarksi”

Graph Pebbling



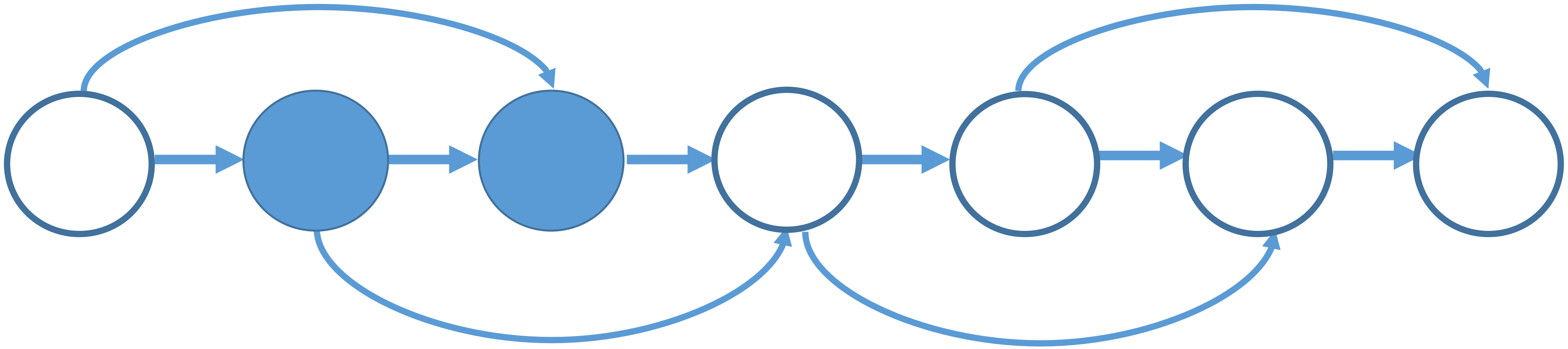
$|P_1| = 1$

Graph Pebbling



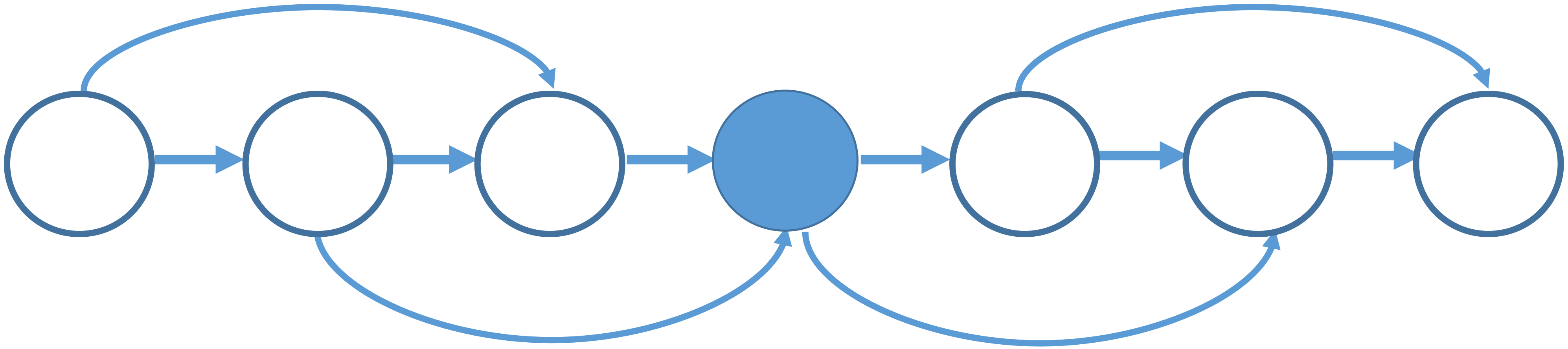
$|P_1| + |P_2| = 1 + 2 = 3$

Graph Pebbling



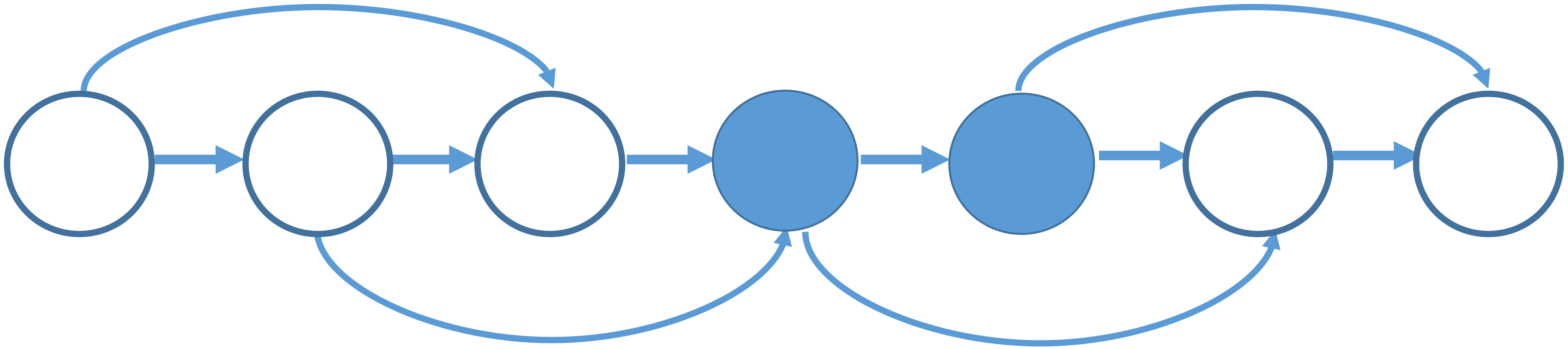
$$|P_1| + |P_2| + |P_3| = 3 + 2 = 5$$

Graph Pebbling



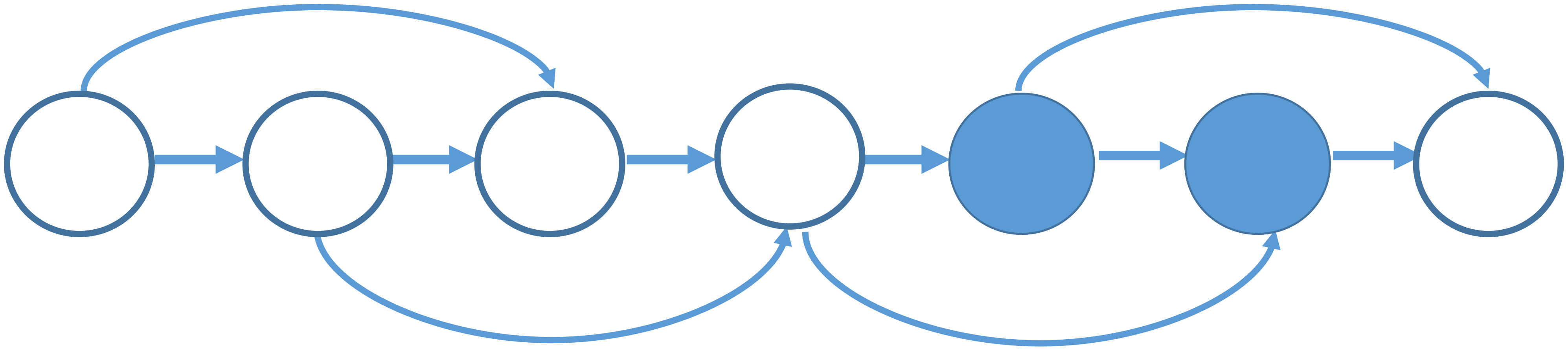
$$\sum_{\{i=1\}}^{4} |P_i| = 5 + 1 = 6$$

Graph Pebbling



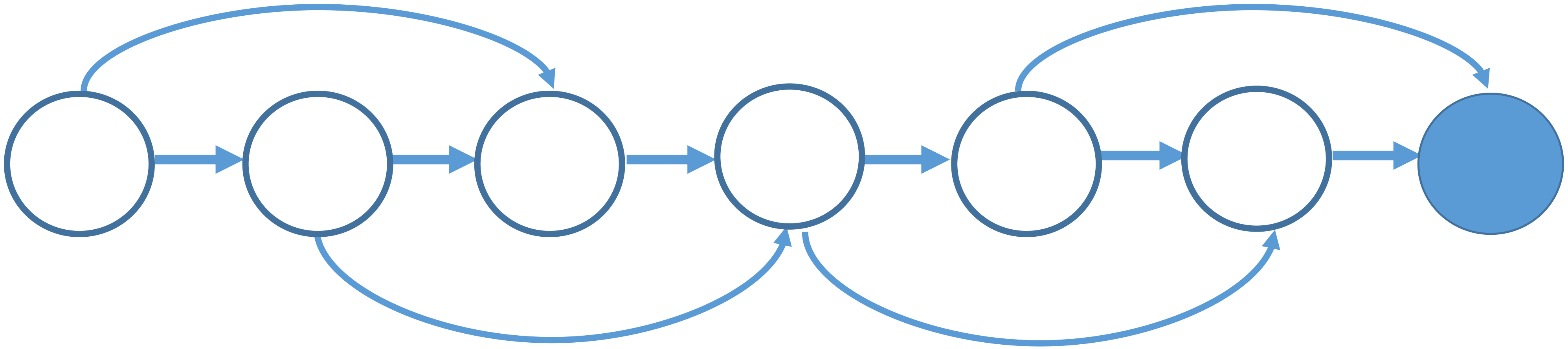
$$\sum_{\{i=1\}}^{5} |P_i| = 6 + 2 = 8$$

Graph Pebbling



$$\sum_{\{i=1\}}^{6} |P_i| = 8 + 2 = 10$$

Graph Pebbling



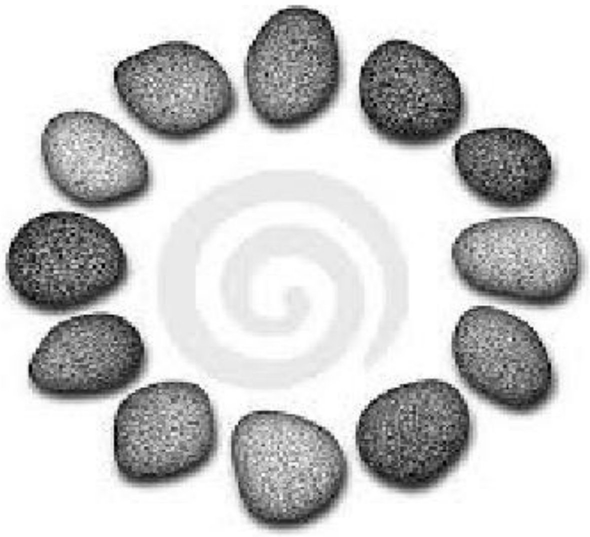
$$cc(G) = \sum_{\{i=1\}}^{7} |P_i| = 10 + 1 = 11$$

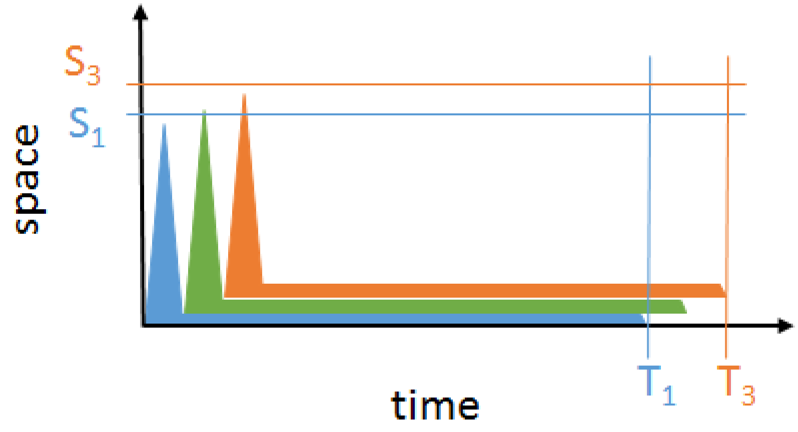
iMHFs

- [AS15] CC amortizes well:
  - $CC(n\ copies\ of\ f) = n \times CC(f)$

iMHFs

- Data-independent memory hard functions require comparatively more resources for adversaries to compute
- Calculating an iMHF can be modeled as graph pebbling
- Cumulative complexity better model than space-time complexity





Structure of Talk

- ✓ Background
- ❖ Graph Pebbling
- ❖ “Graph Reducibility”
- ❖ Open Problems

Main Result

❖ Computing $cc(G)$ is NP-hard!

❖ Finding the pebbling number of a graph is PSPACE-complete. [GLT79]

3-PARTITION

- Given set of $3n$ integers, can we partition them into n sets, each with the same sum?
- $\{1,2,4,5,6,7,8,11,13\}$
  - $\{2,4,13\} \rightarrow 19$ $\{1,7,11\} \rightarrow 19$ $\{5,6,8\} \rightarrow 19$
- $\{1,2,3,4,6,7,9,10,11\}$

Bounded 2-Linear Covering

- Given n variables $x_1, x_2, \ldots, x_n$, integers $m \leq k$, and k equations of the form $x_i + c = x_j$, can we find m assignments so that all equations are satisfied?
- $x_1 + 2 = x_2,\quad x_2 + 3 = x_3,\quad x_1 + 6 = x_3$
- $x_1 + 5 = x_2,\quad x_2 + 1 = x_3,\quad x_1 + 5 = x_3$
- $m = 2$
- Assignment 1: $x_1 = 1, x_2 = 3, x_3 = 6$
- Assignment 2: $x_1 = 1, x_2 = 6, x_3 = 7$

Reductions

- Reducing 3-PARTITION to B2LC
- Reducing B2LC to $cc(G)$

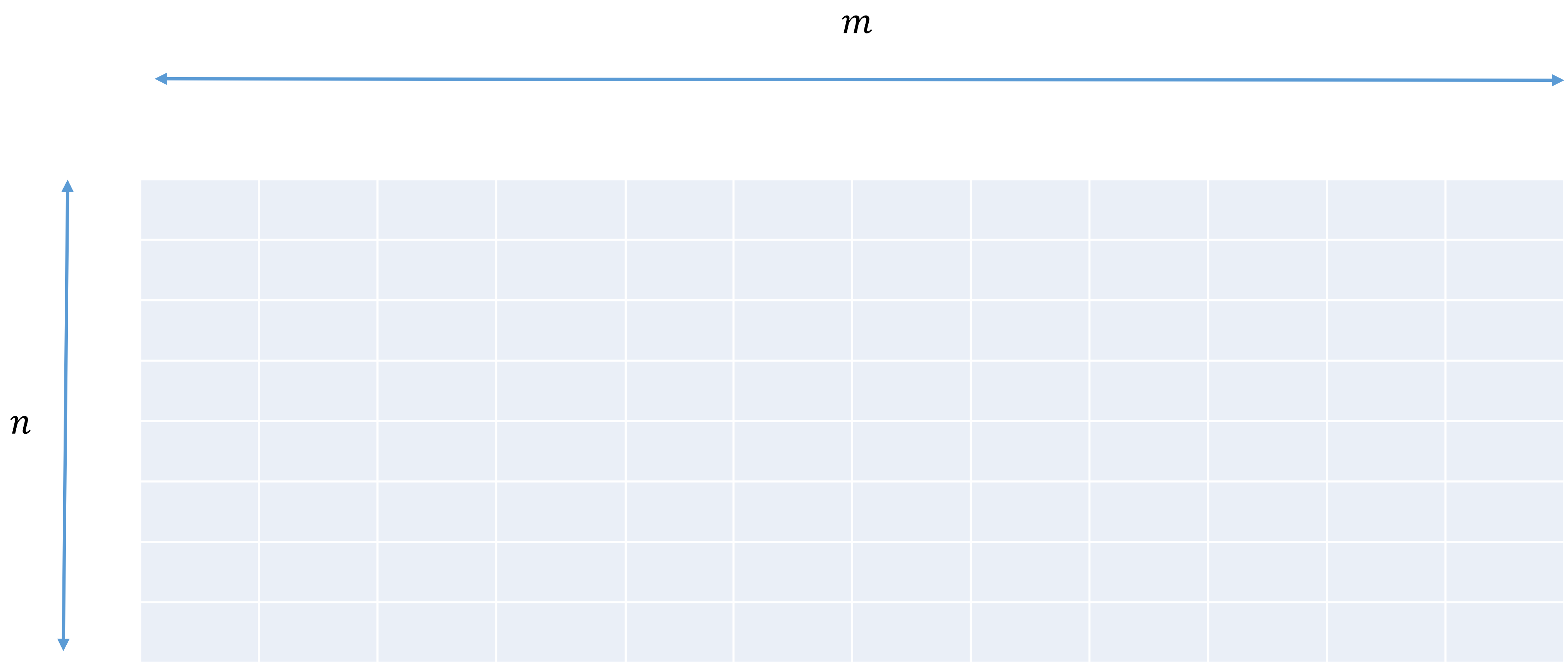
Reductions

$$T = \sum_{\{i=1\}}^{m} s_m$$

$$
\begin{aligned}
x_1 + s_1 = x_2, \quad & x_2 + s_2 = x_3, \quad \ldots, \quad x_m + s_m = x_{m+1}, \\
x_1 + 0 = x_2, \quad & x_2 + 0 = x_3, \quad \ldots, \quad x_m + 0 = x_{m+1}, \\
x_1 + T = x_2, \quad & x_2 + T = x_3, \quad \ldots, \quad x_m + T = x_{m+1}, \\
x_1 + 2T = x_2, \quad & x_2 + 2T = x_3, \quad \ldots, \quad x_m + 2T = x_{m+1}, \\
x_1 + (n-2)T = x_2, \quad & x_2 + (n-2)T = x_3, \quad \ldots, \quad x_m + (n-2)T = x_{m+1},
\end{aligned}
$$

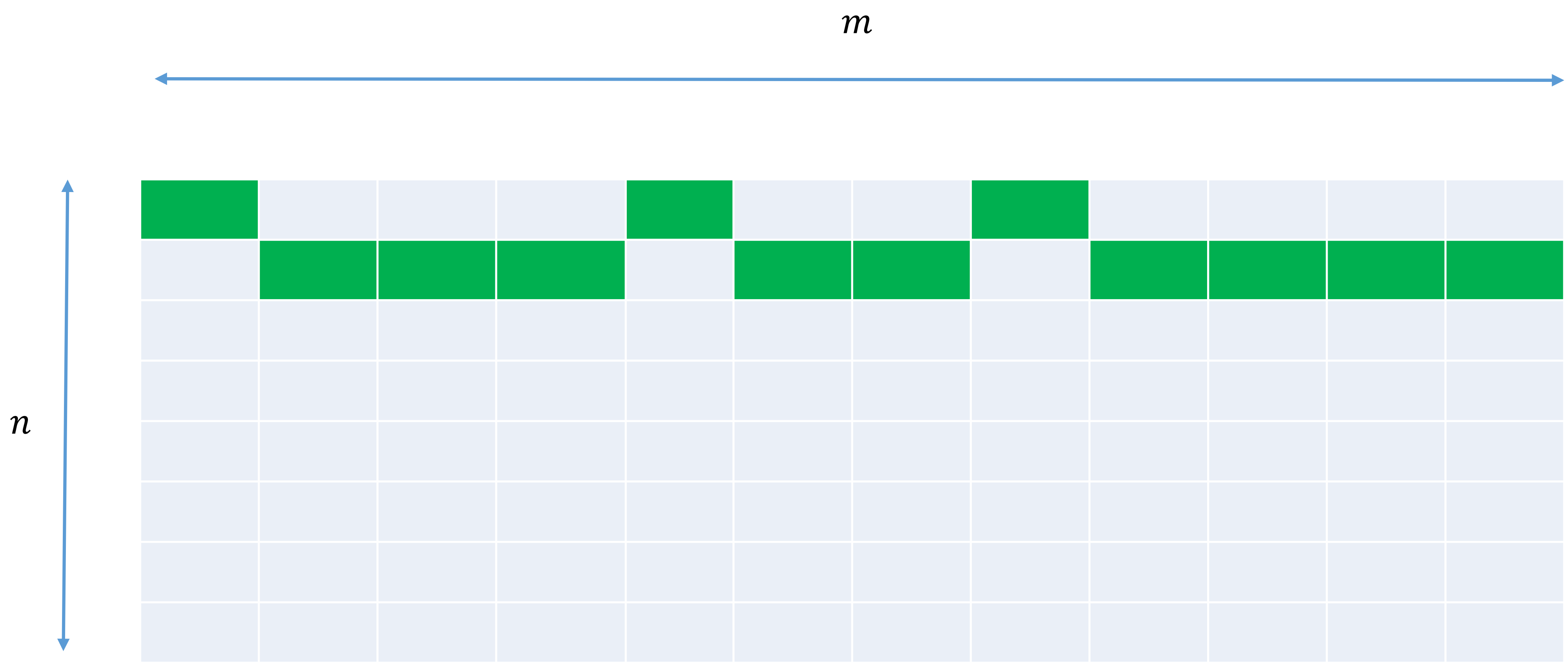
$$x_1 + \frac{T}{n} + 3(i-1)(n-2)T = x_{m+1},$$

Reductions



$m = 3n$ integers in original 3-PARTITION instance

Reductions



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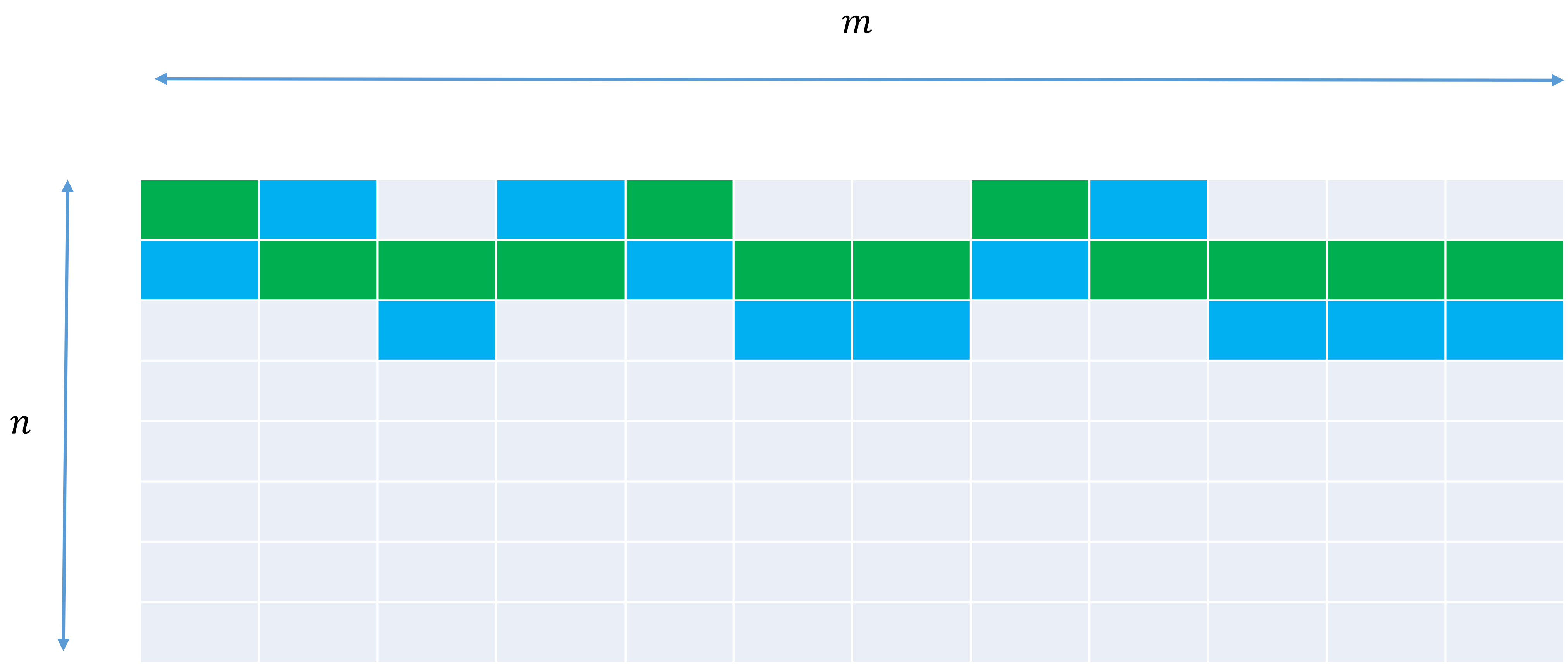
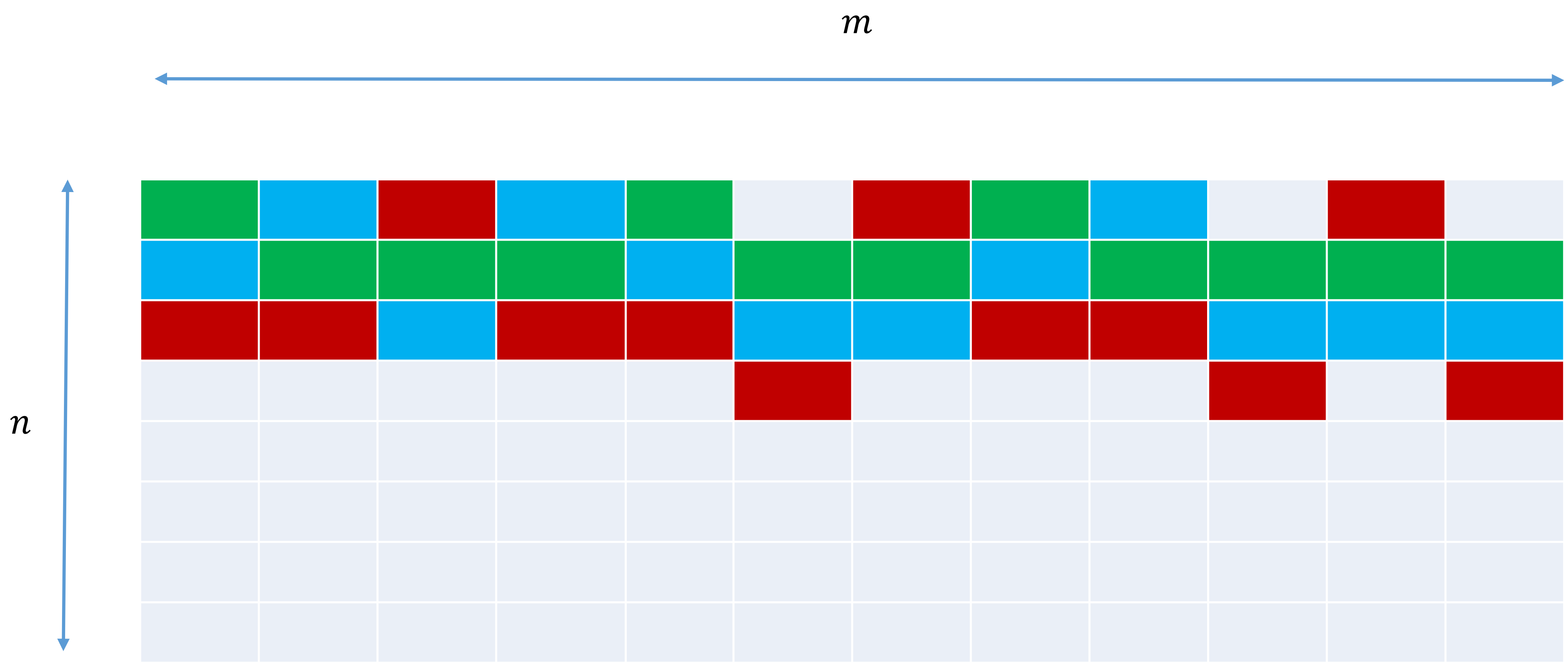
Reductions



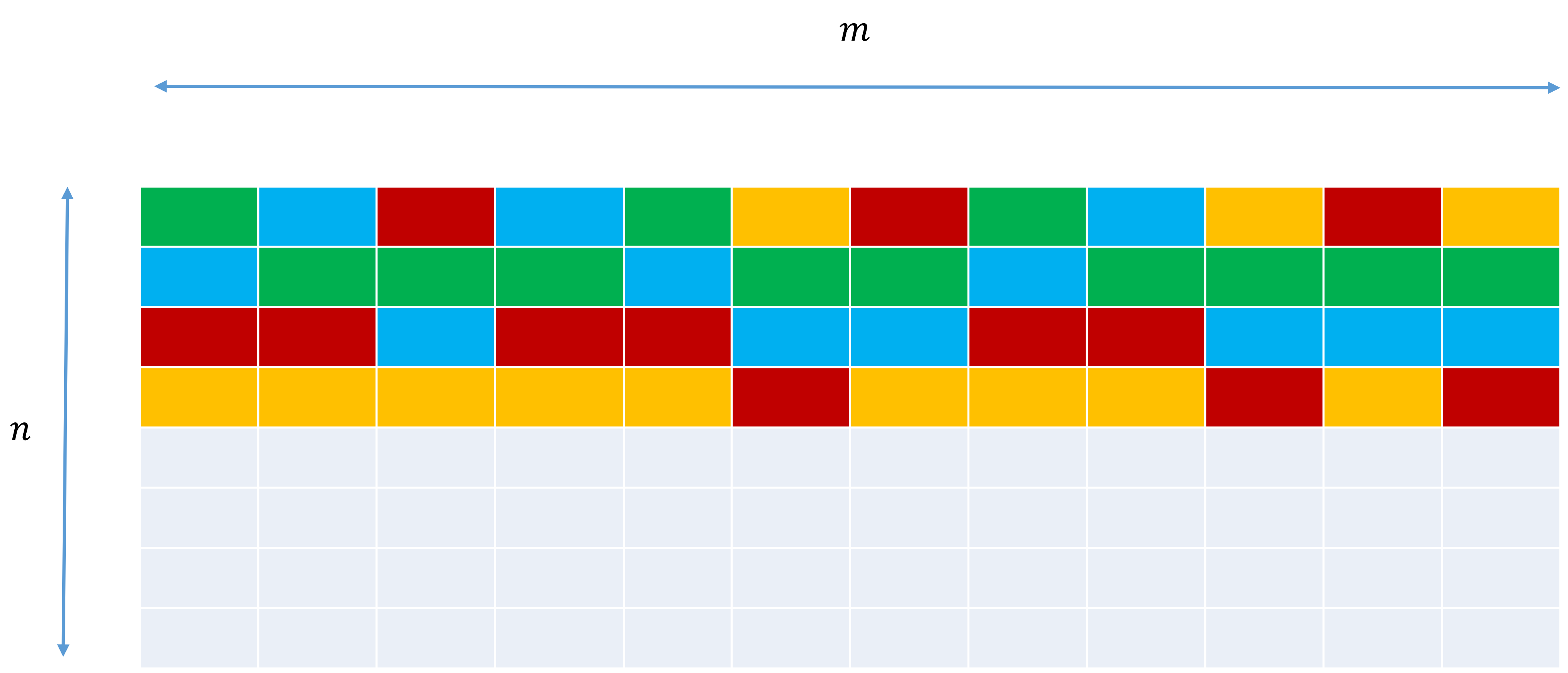
$m = 3n$ integers in original 3-PARTITION instance

Reductions



$m = 3n$ integers in original 3-PARTITION instance

Reductions



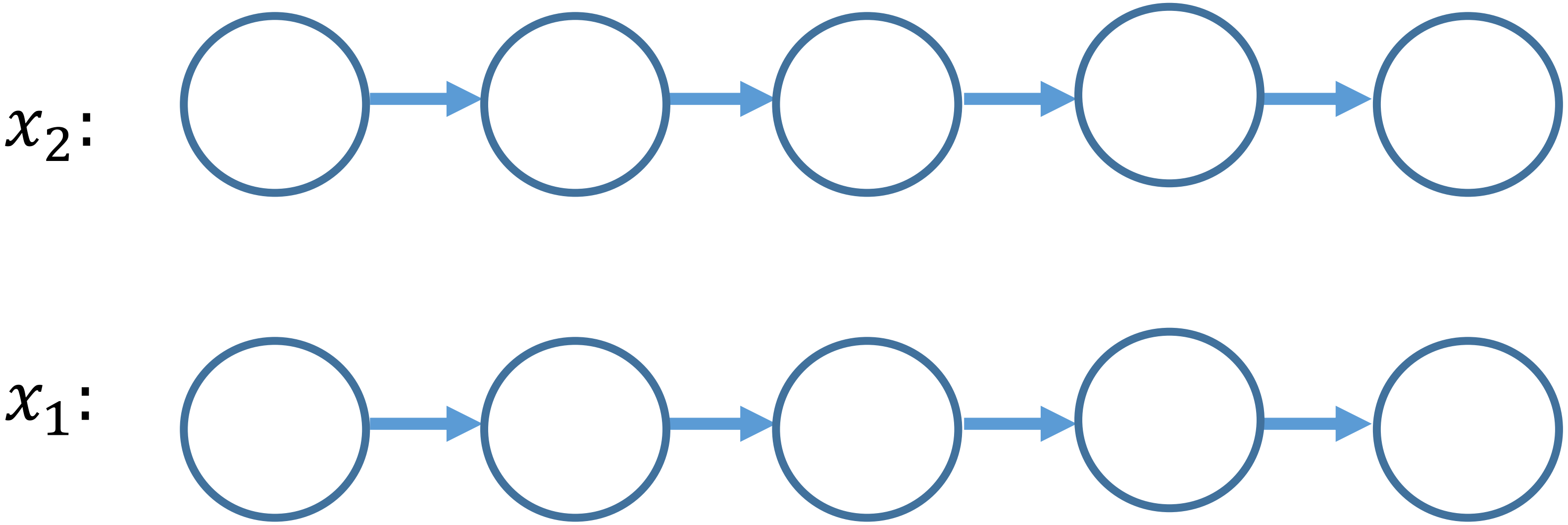
$m = 3n$ integers in original 3-PARTITION instance

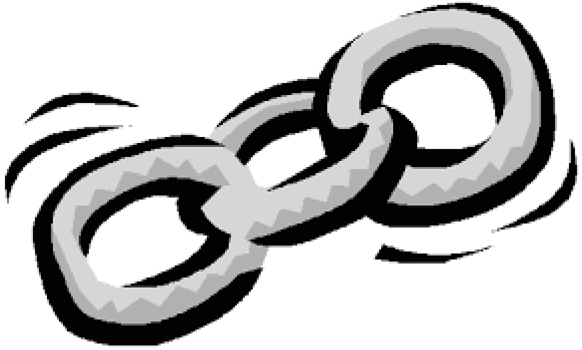
Reductions

✓ Reducing 3-PARTITION to B2LC

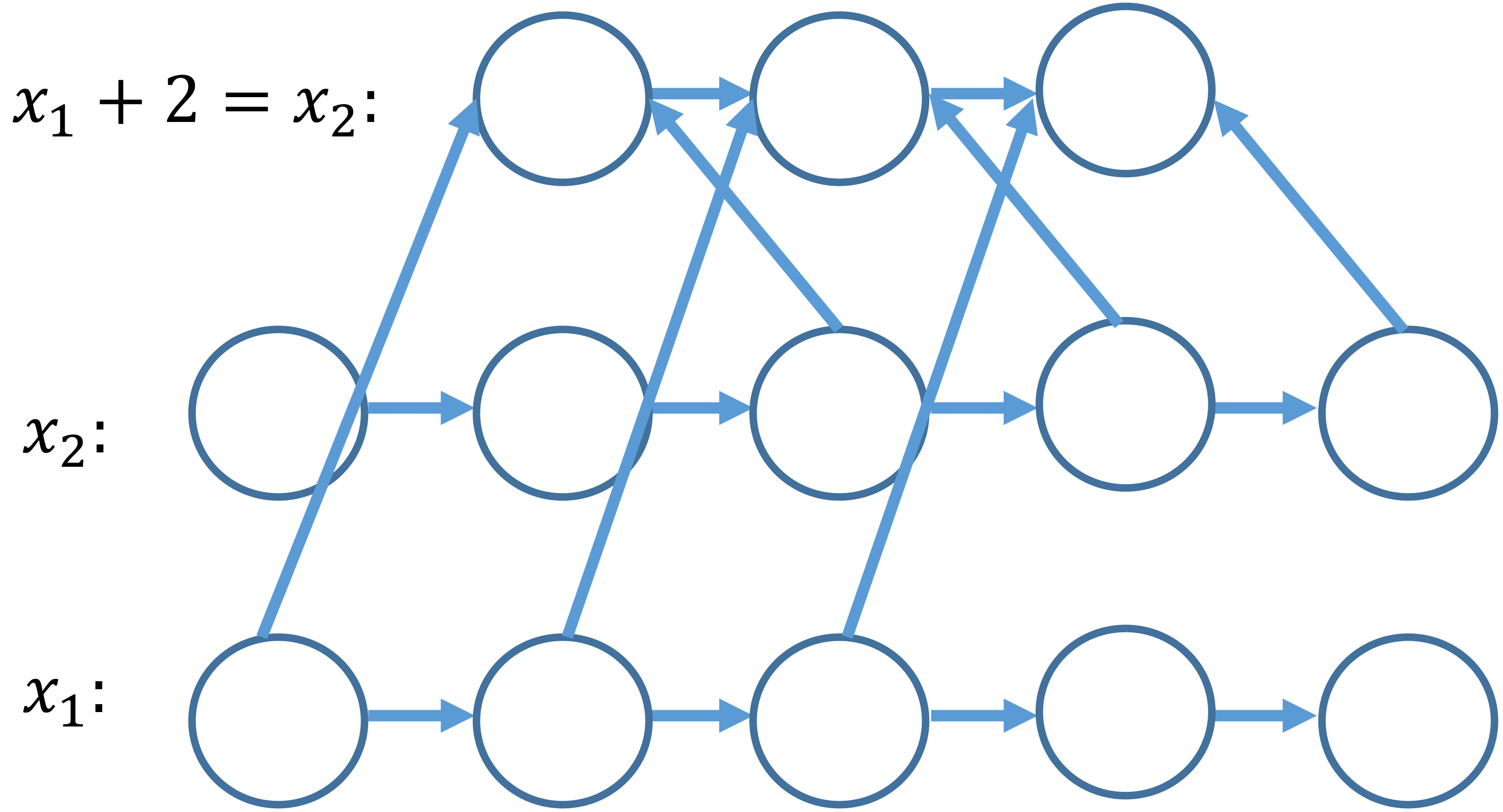
❖ Reducing B2LC to $cc(G)$

Reductions

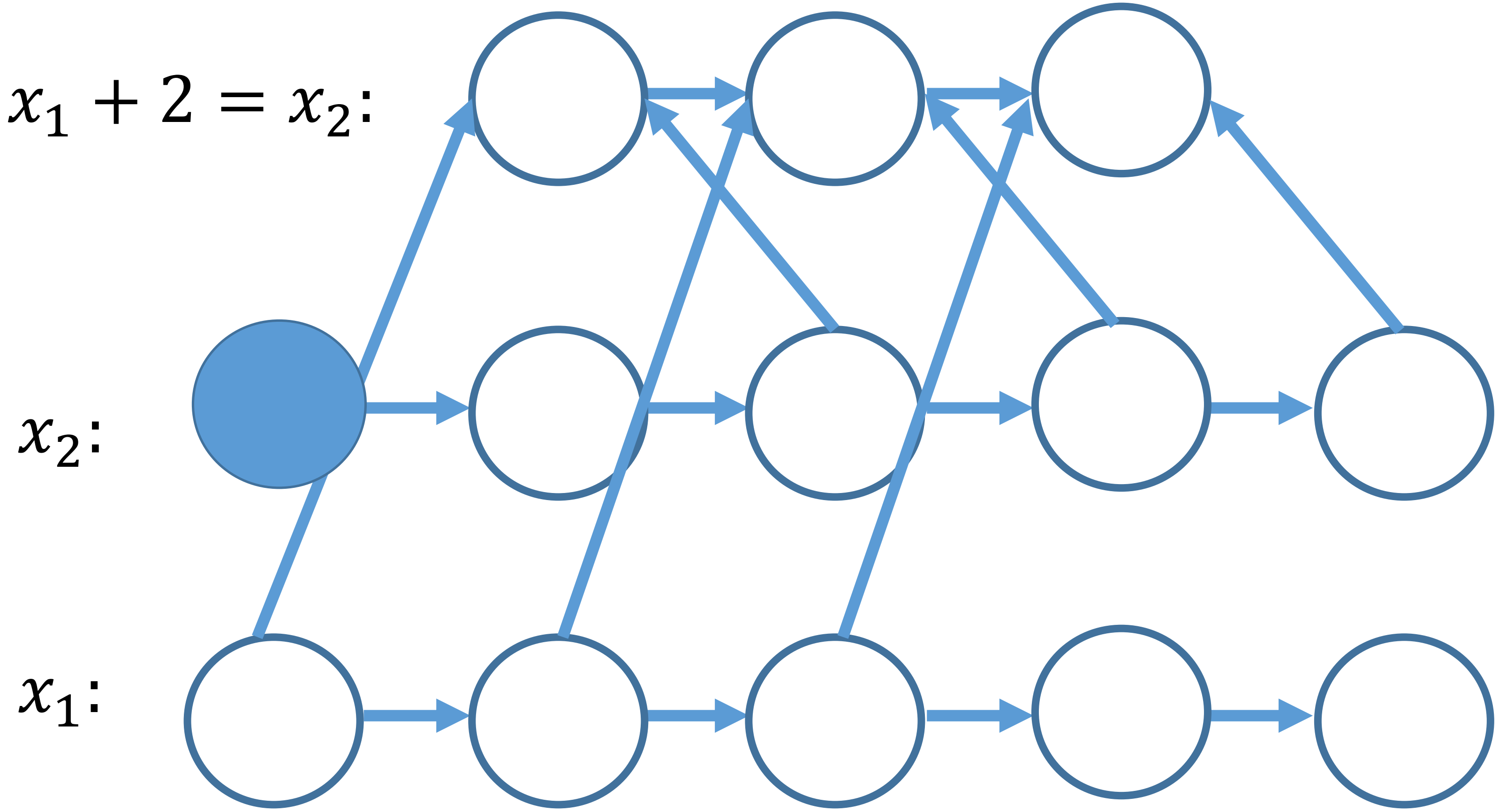




Reductions



Honest Pebbling



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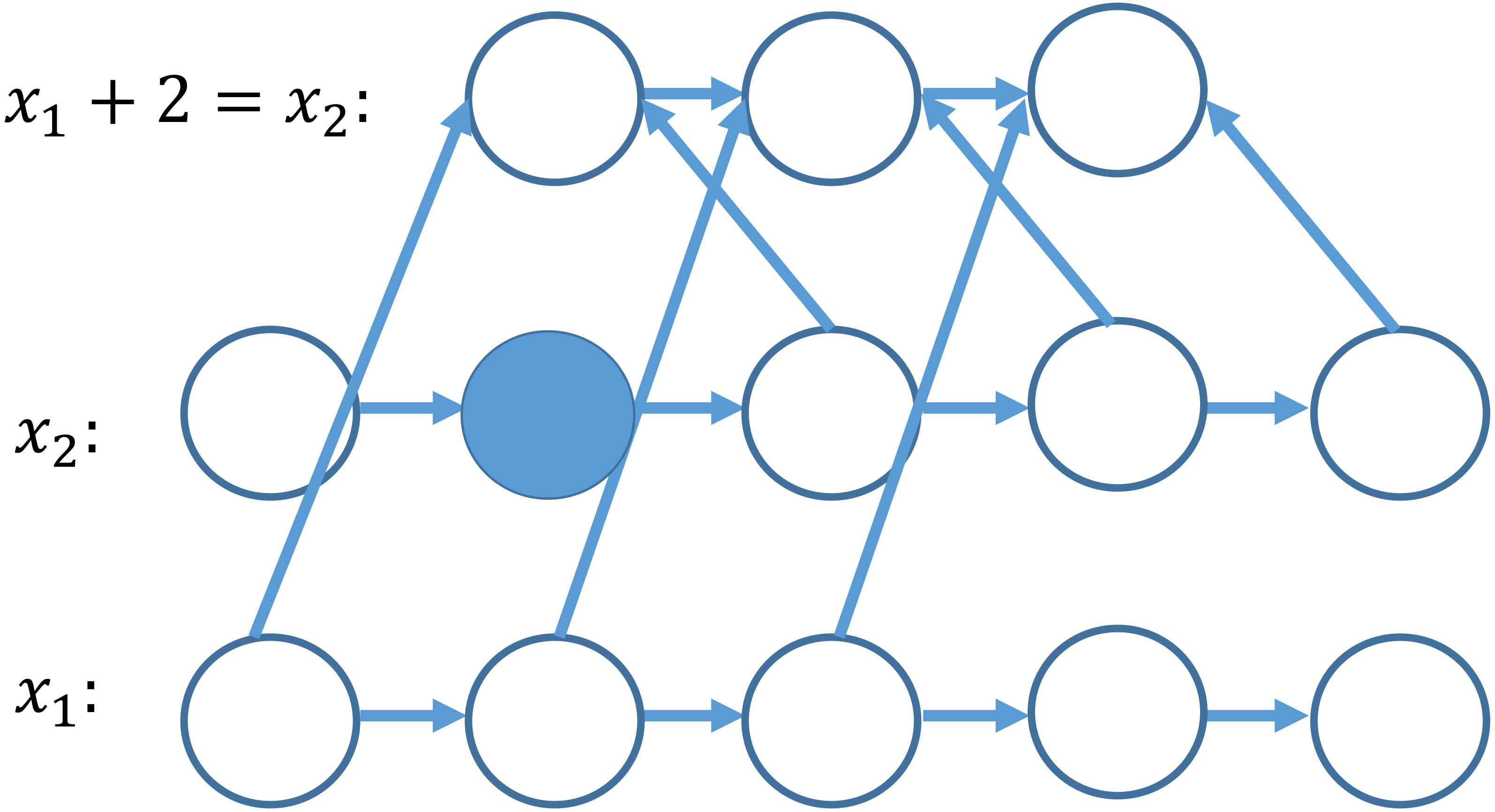

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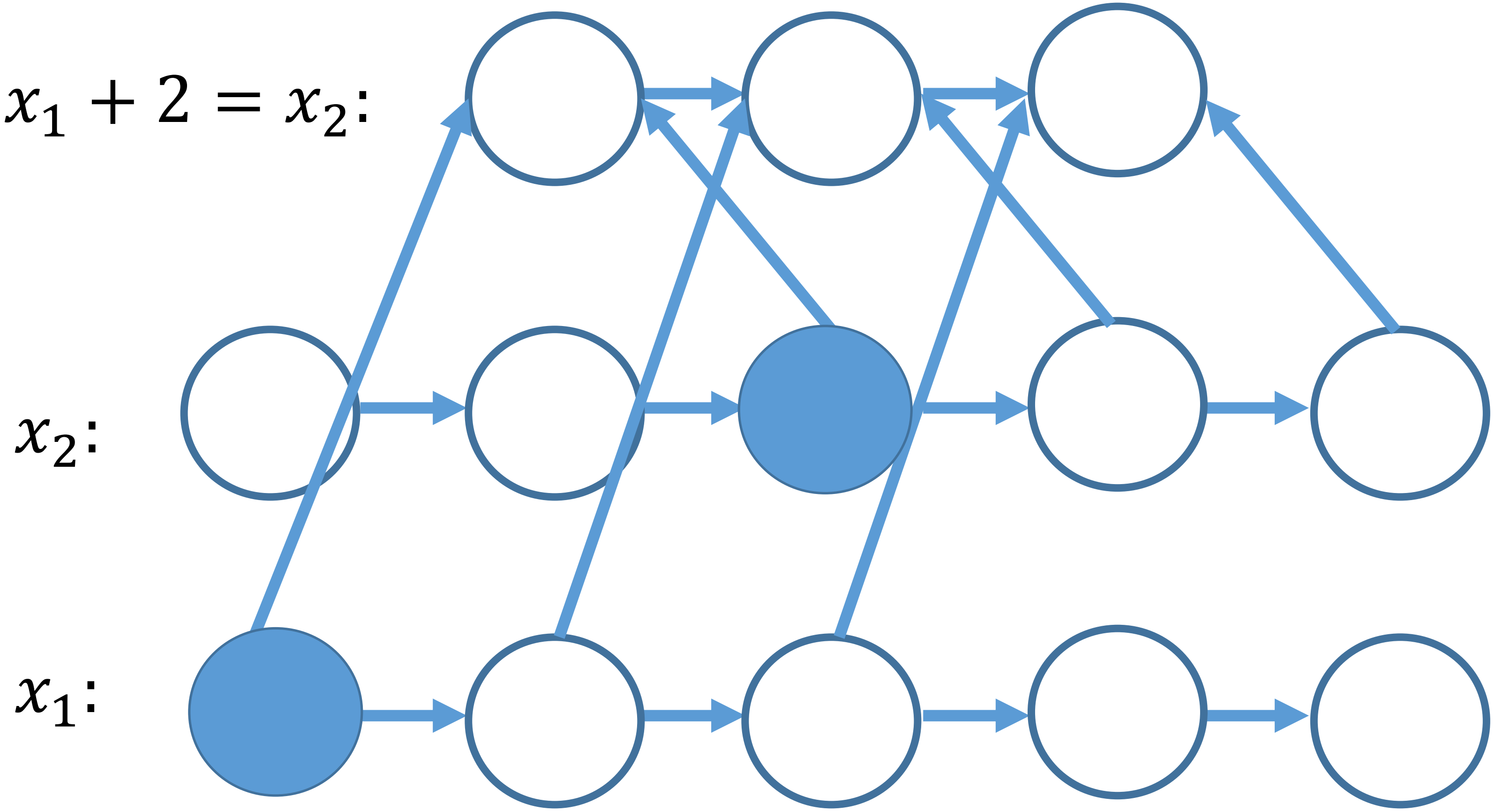

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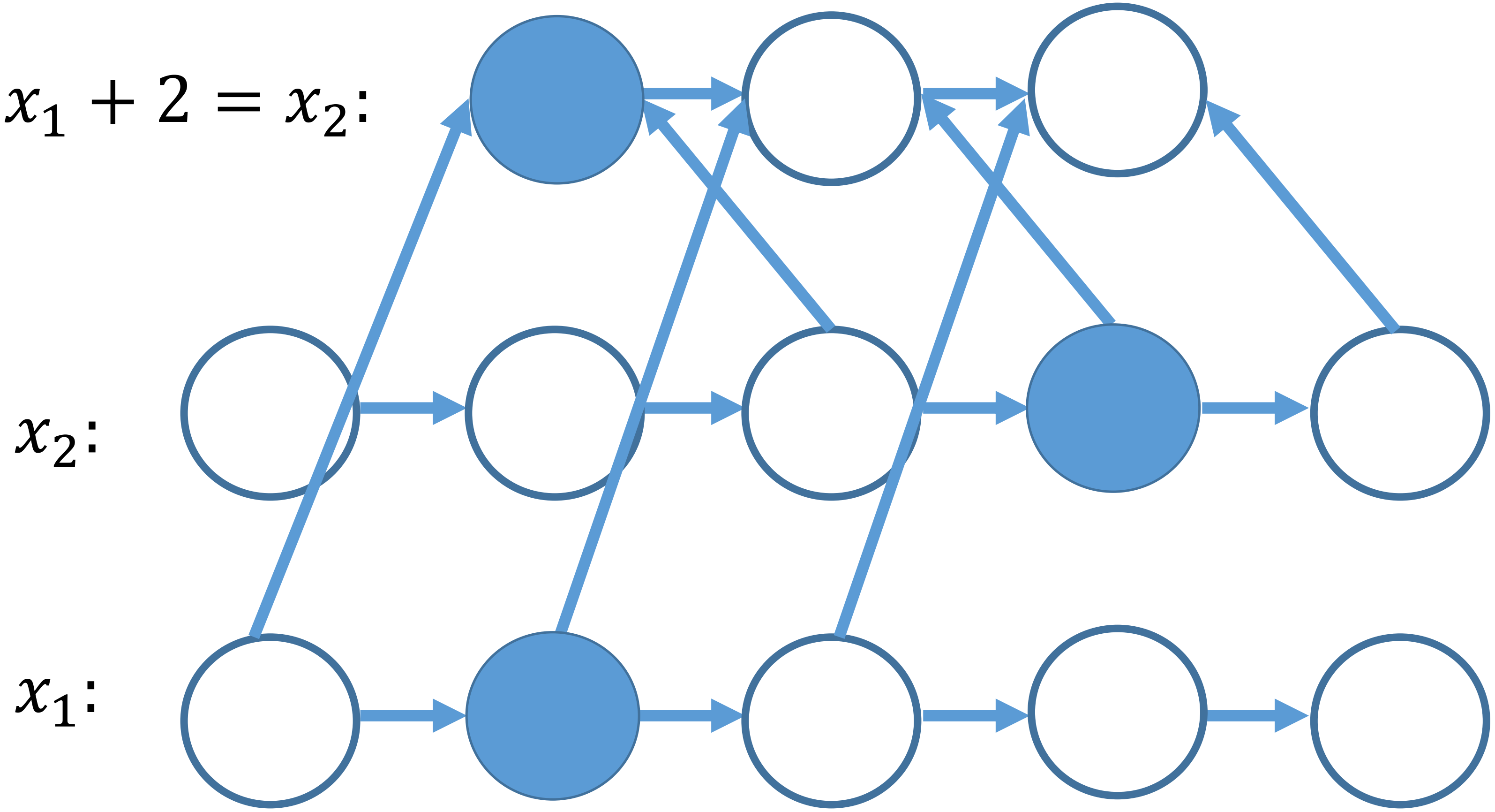

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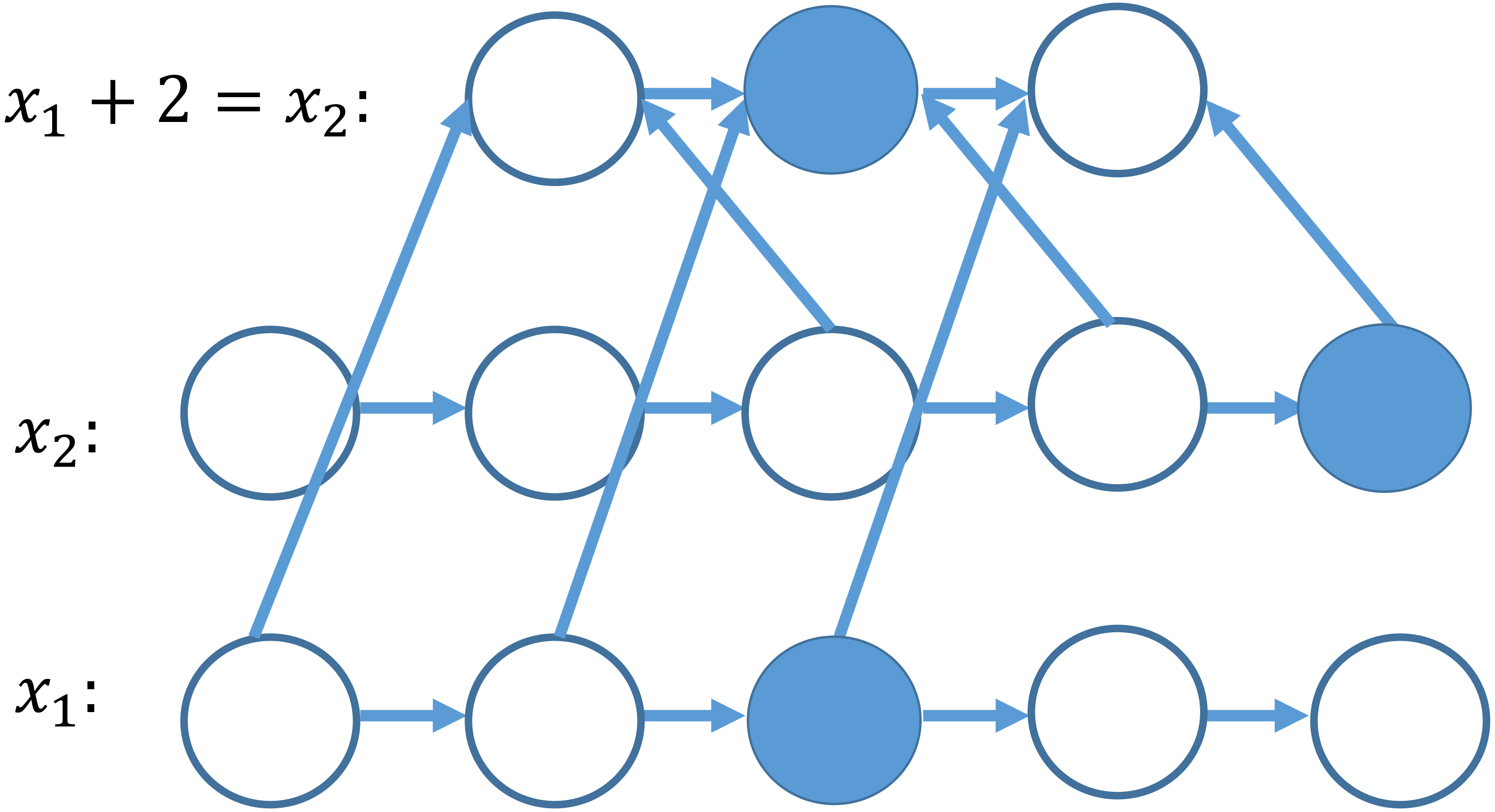

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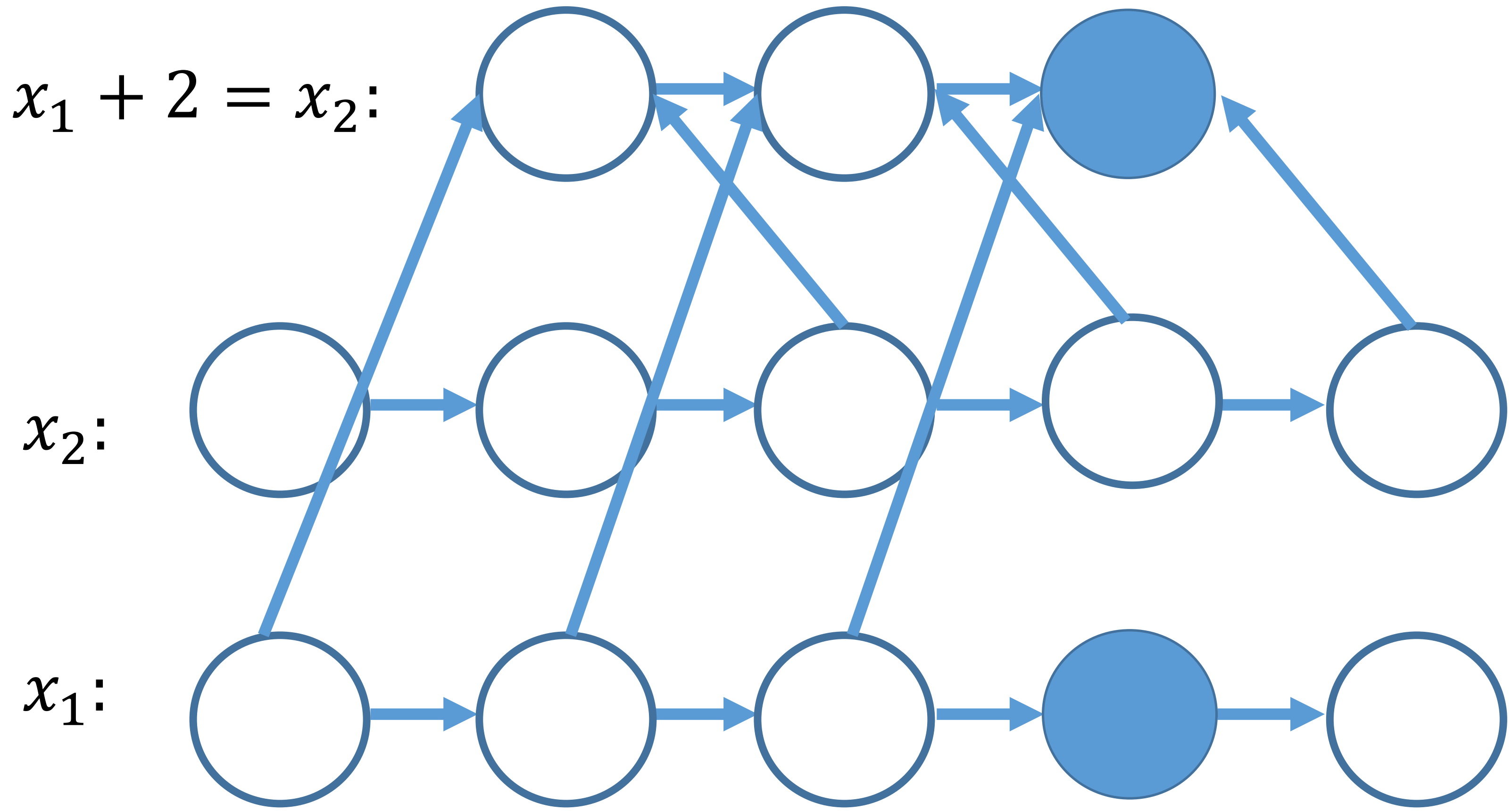

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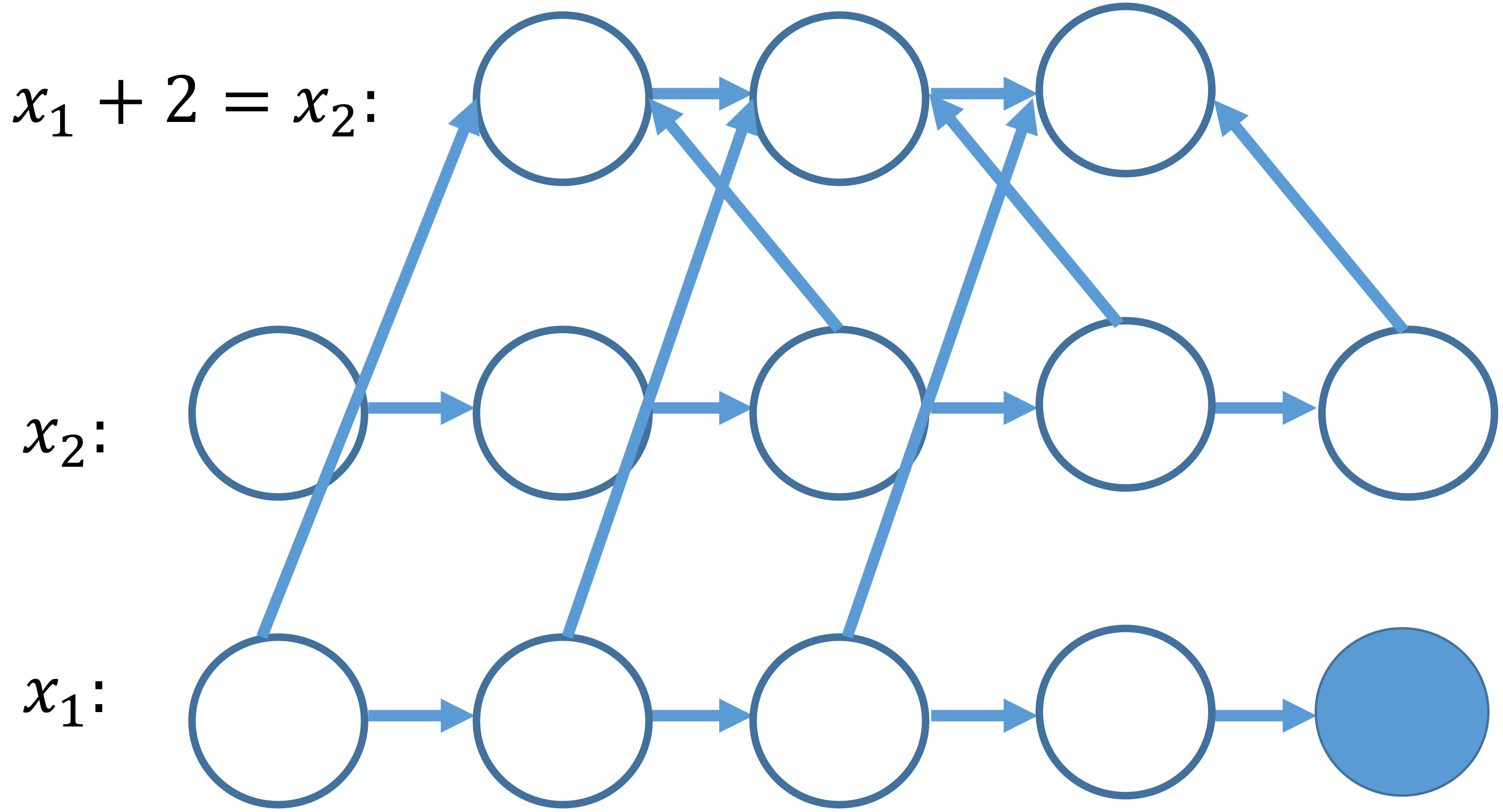
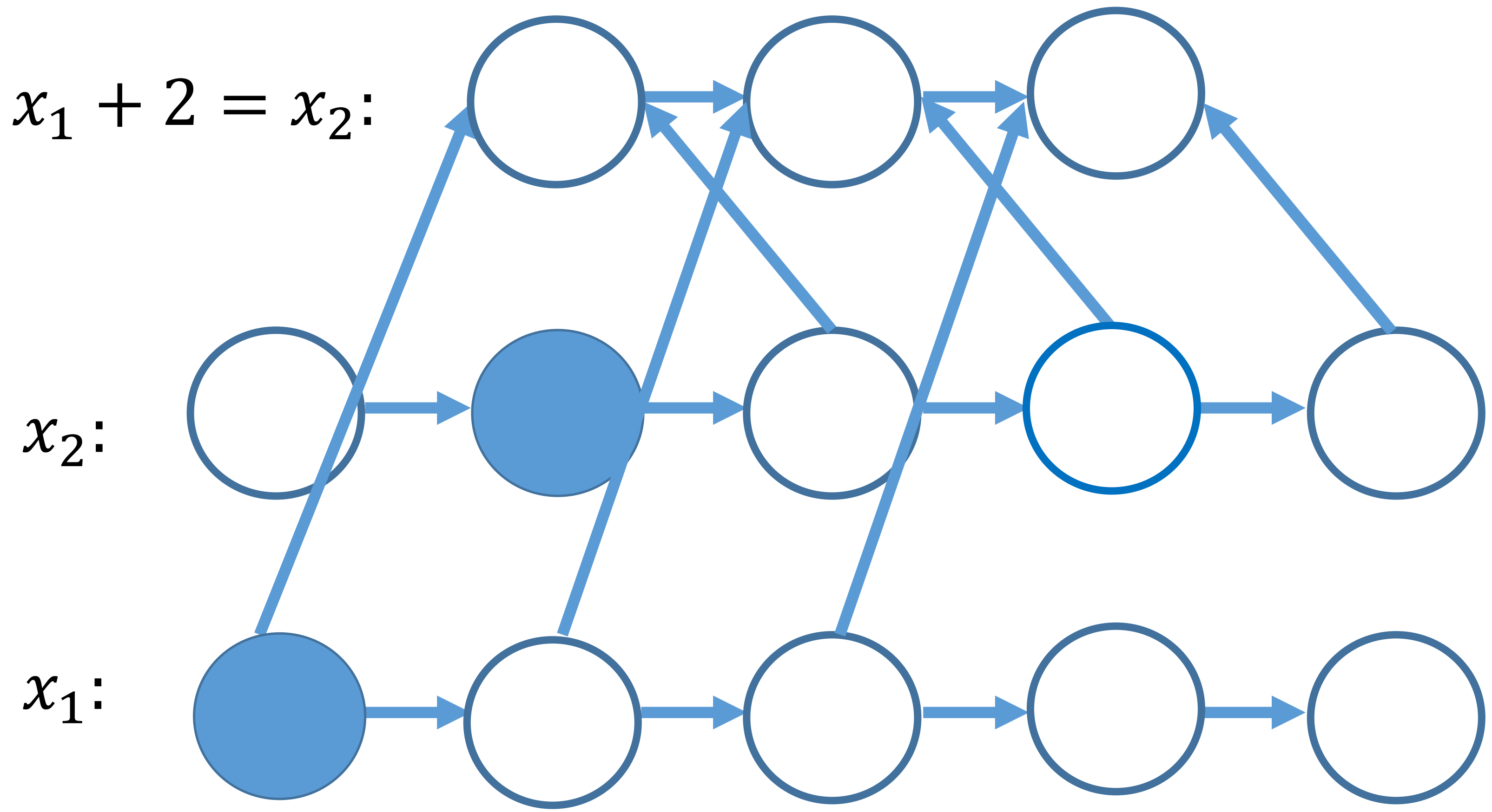


Cheater!



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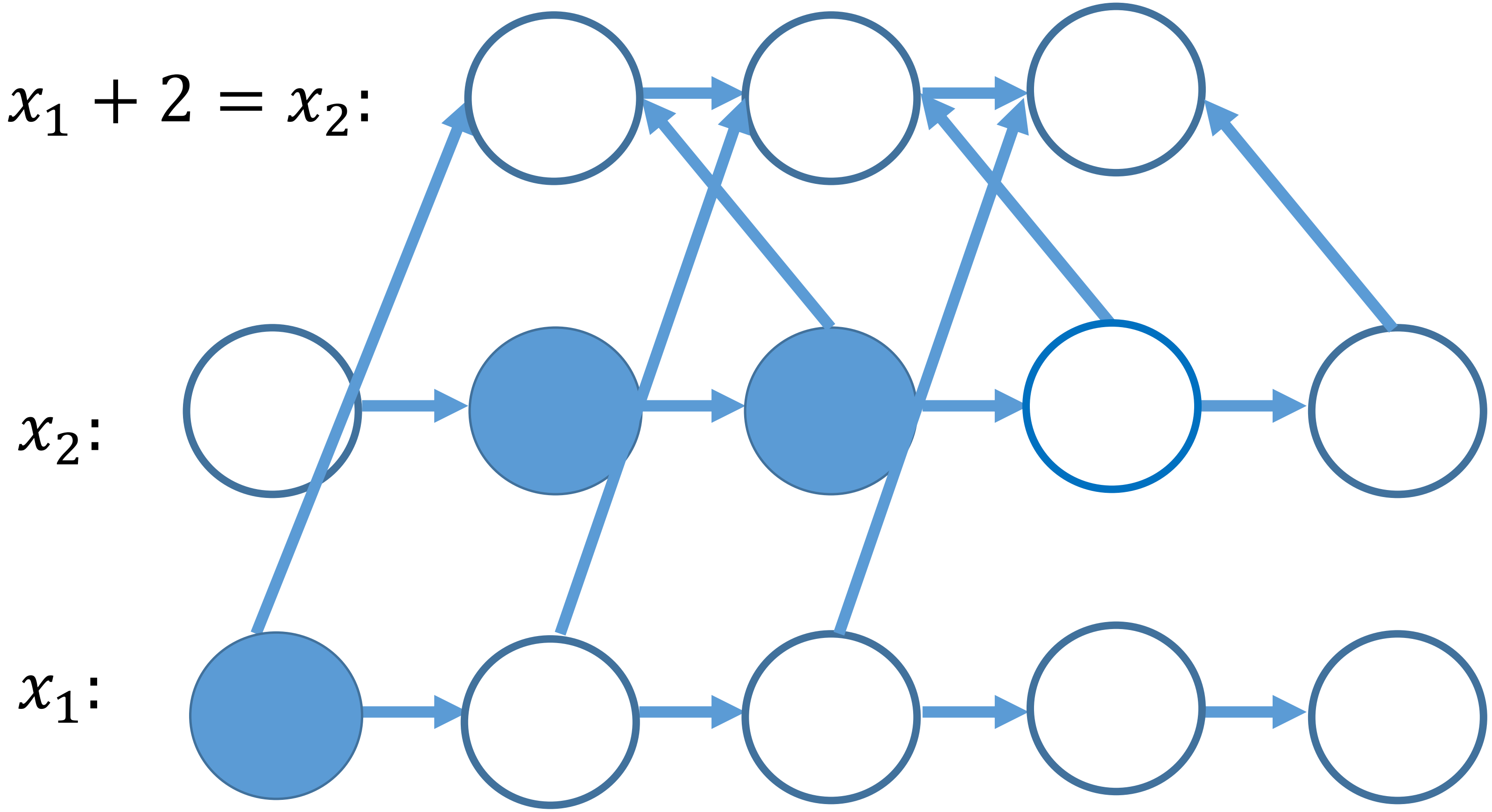

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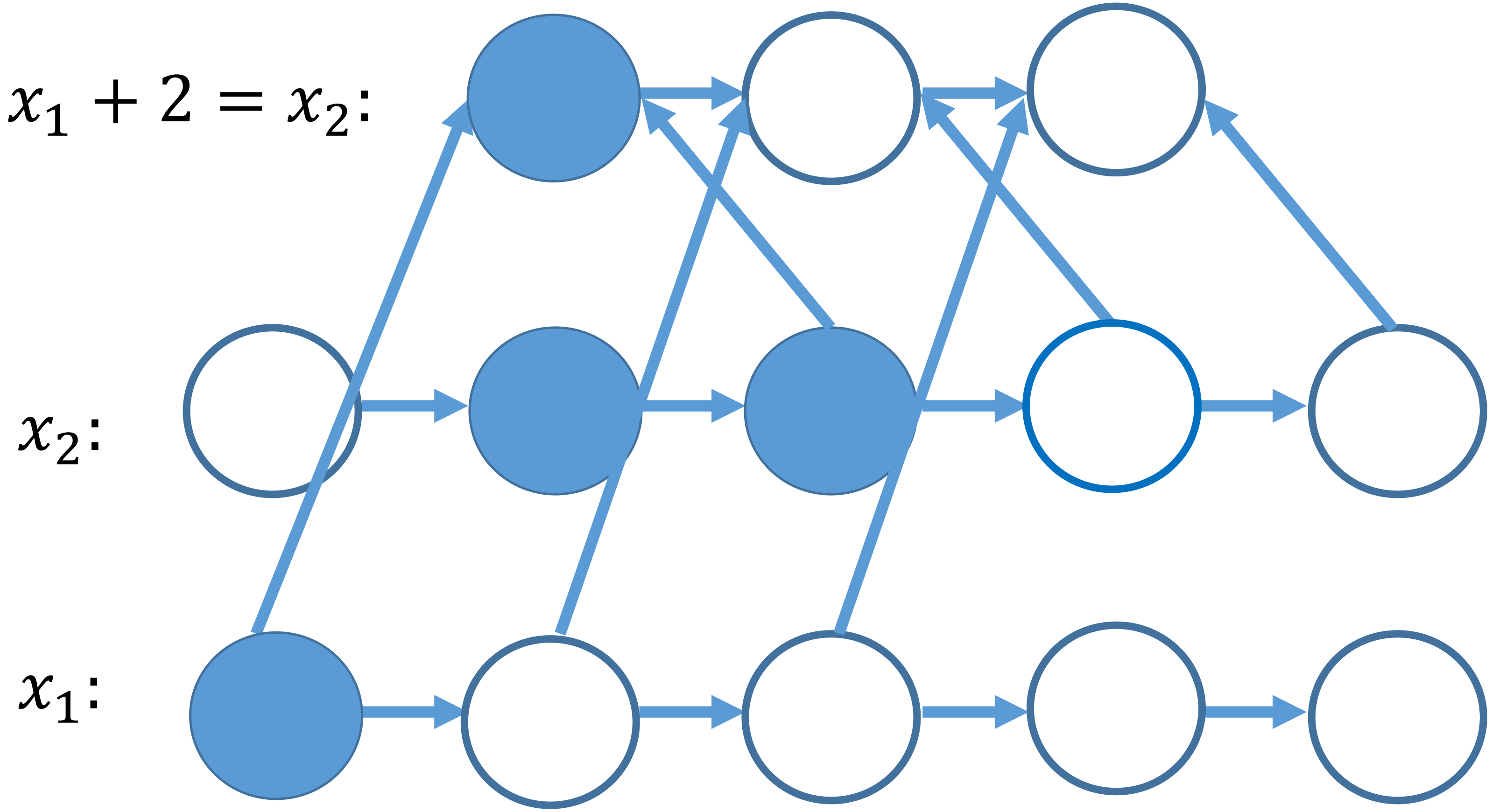

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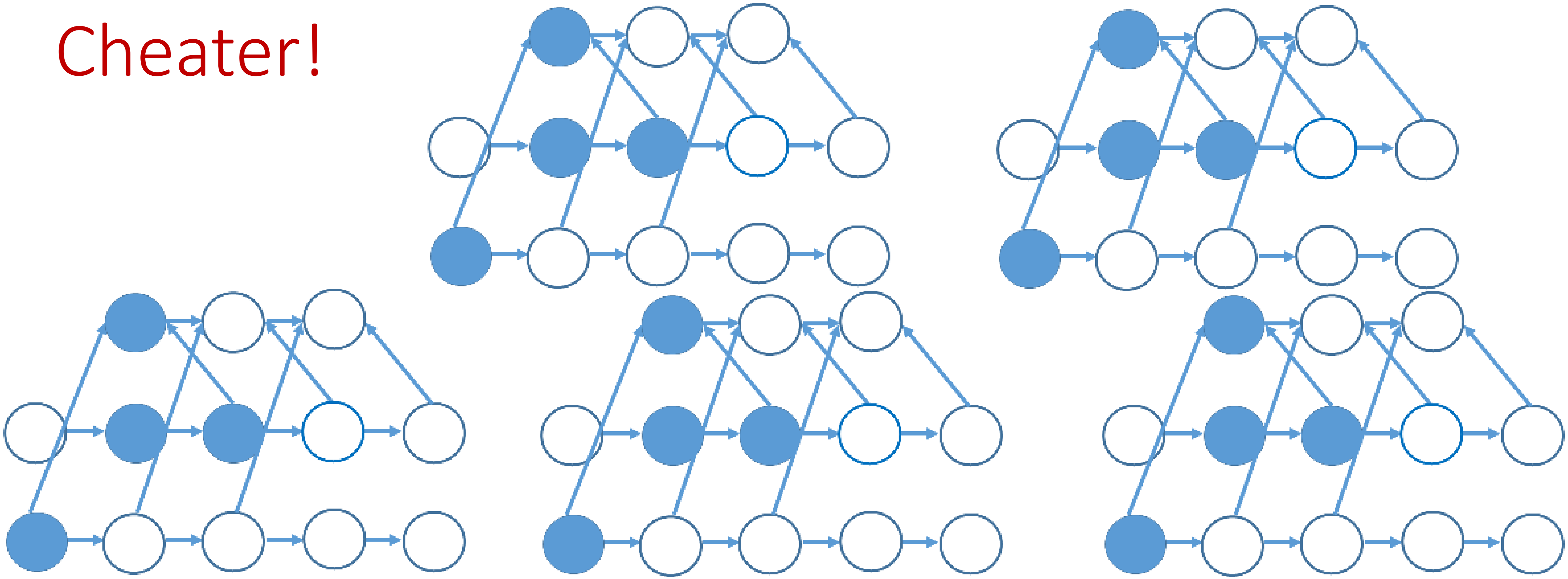


Lemma 4 *If the* B2LC *instance has a valid solution, then* $\Pi^{\|}_{cc}(G_{B2LC}) \le \tau cmn + 2cmn + 2ckm + 1$.

Lemma 5 *If the* B2LC *instance does not have a valid solution, then* $\Pi^{\|}_{cc}(G_{B2LC}) \ge \tau cmn + \tau$.

Reductions

Figure 5 shows an example of a reduction in its entirety when $\tau = 1$.

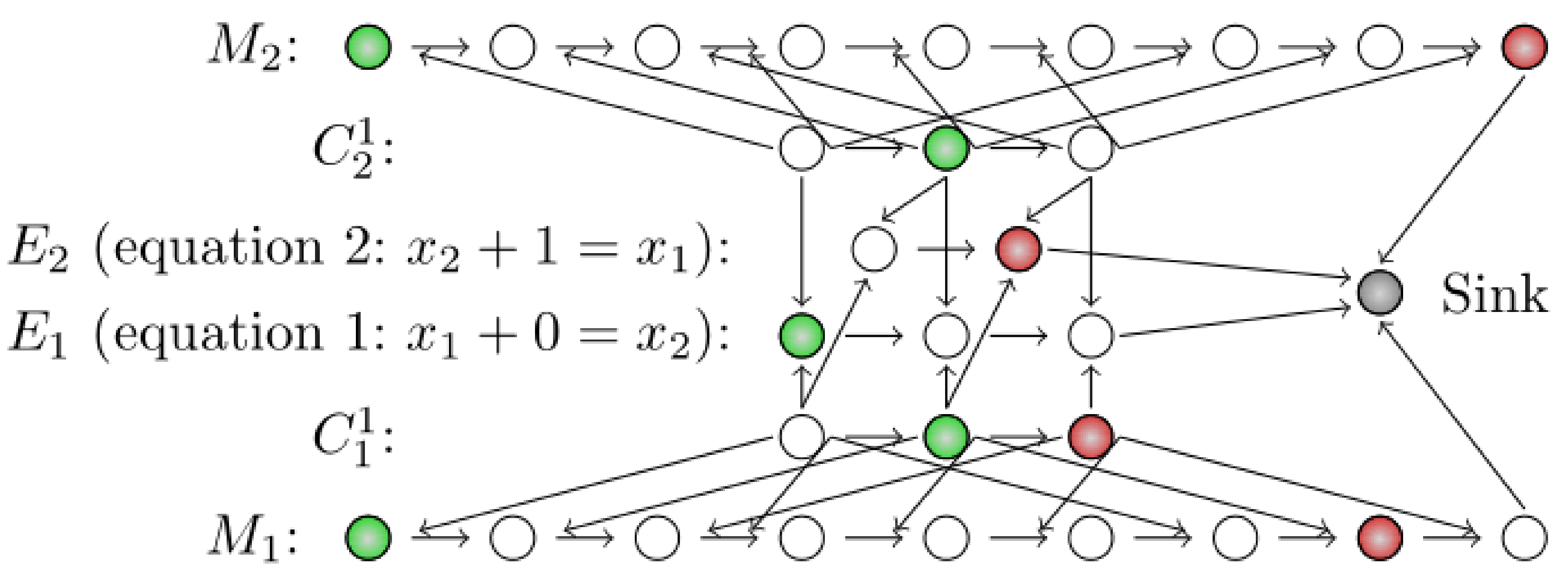


Fig. 5. An example of a complete reduction G_{B2LC}, again $m = 3$ and $c = 3$. The green nodes represent the pebbled vertices at time step 2 while the red nodes represent the pebbled vertices at time step 10.

Structure of Talk

- ✓ Background
- ✓ Graph Pebbling
- ❖ “Graph Reducibility”
- ❖ Open Problems

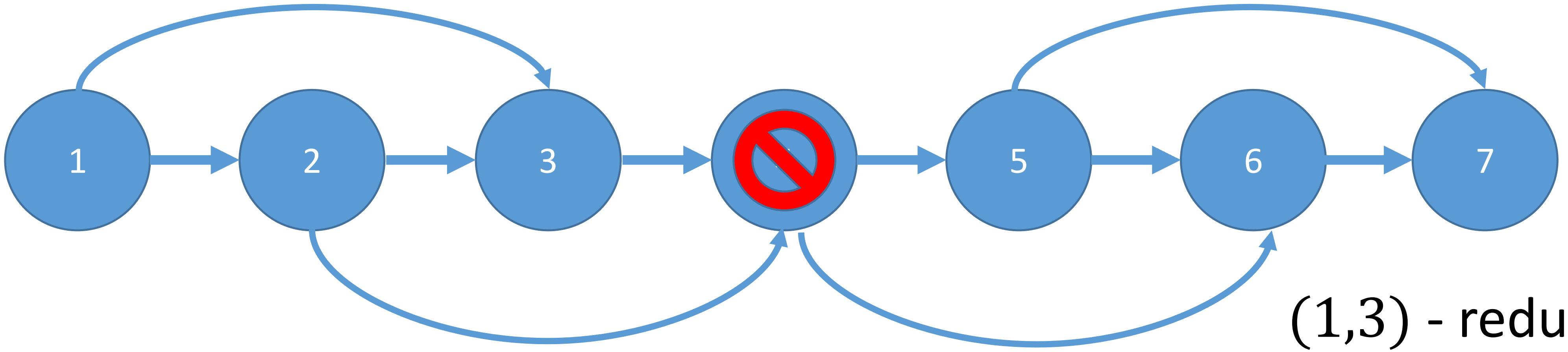
Results

- Computing $cc(G)$ is NP-hard!

- Computing (e, d)-reducibility is NP-hard!

Graph Reducibility

- We say that a directed acyclic graph G is (e, d)-reducible if there exists a set S of e nodes such that $G - S$ has depth at most d.



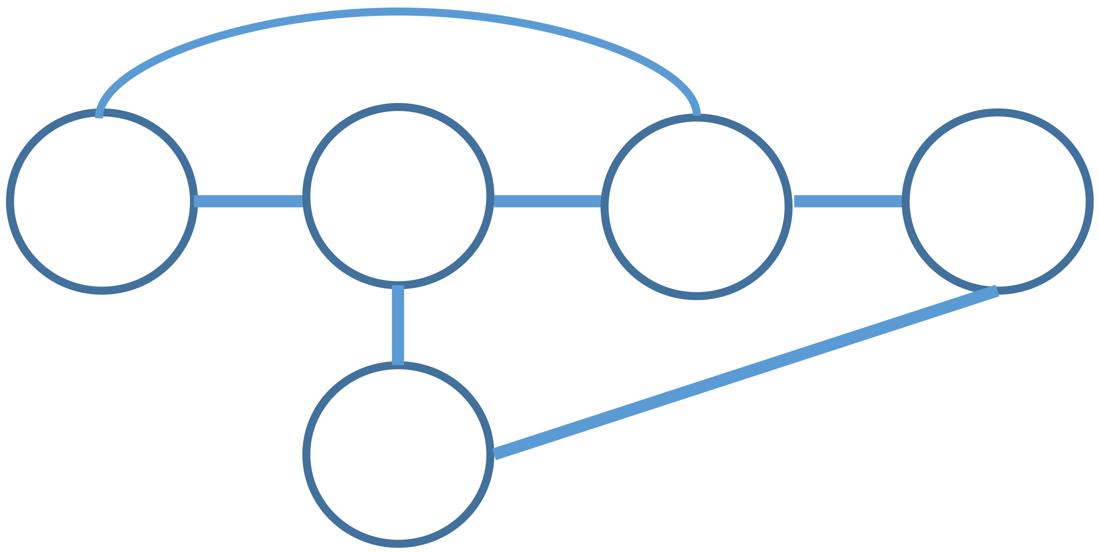
$(1,3)$ - reducible

Graph Reducibility

- Can deduce $cc(G)$ from (e, d)-reducible!
- Depth-robustness is a *necessary* condition for secure iMHFs (AB16)
  - There exists attack with $E_R(A) = O\left(en + \sqrt{n^3 d}\right)$, which is $o(n^2)$ for $e, d = o(n)$.
- Depth-robustness is a *sufficient* condition for secure iMHFs (ABP16)
  - $cc(G) \leq ed$

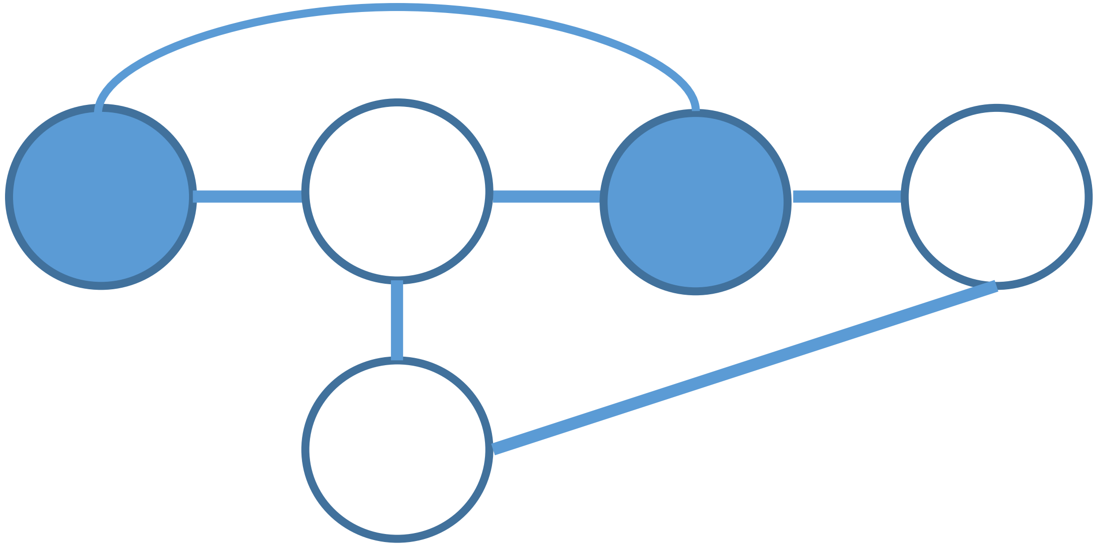
Vertex Cover

- Given a graph $G(V, E)$ and an integer k, does there exist a subset of V of size k which intersects all edges of E?



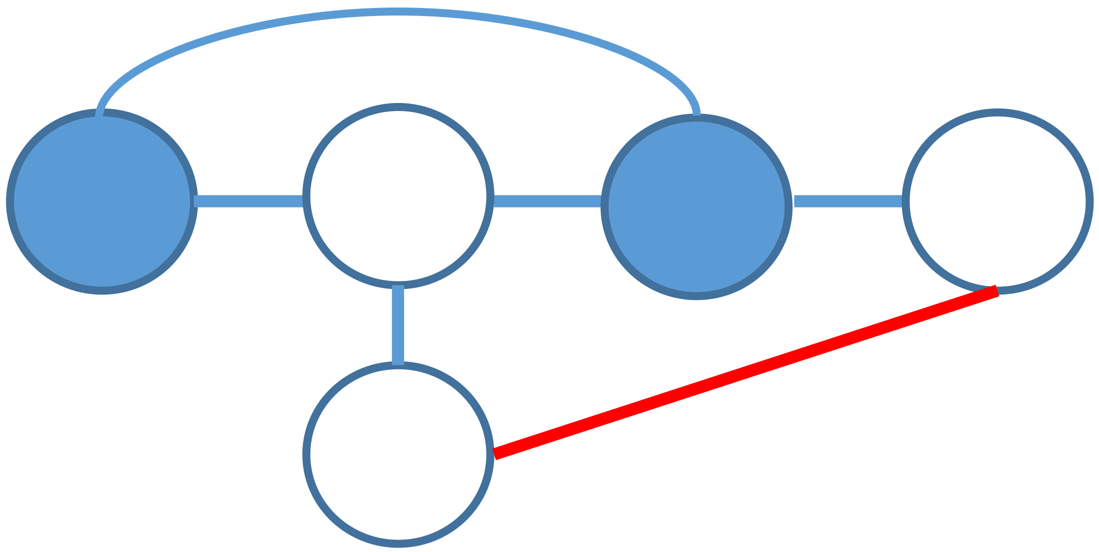
Vertex Cover

❖ Given a graph $G(V,E)$ and an integer k, does there exist a subset of V of size k which intersects all edges of E?



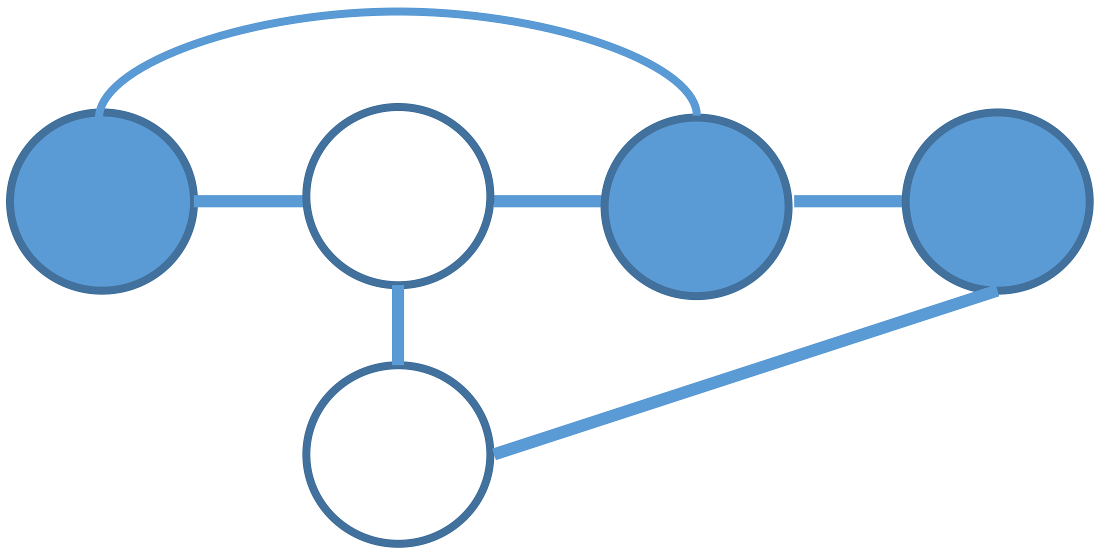
Vertex Cover

❖ Given a graph $G(V,E)$ and an integer k, does there exist a subset of V of size k which intersects all edges of E?



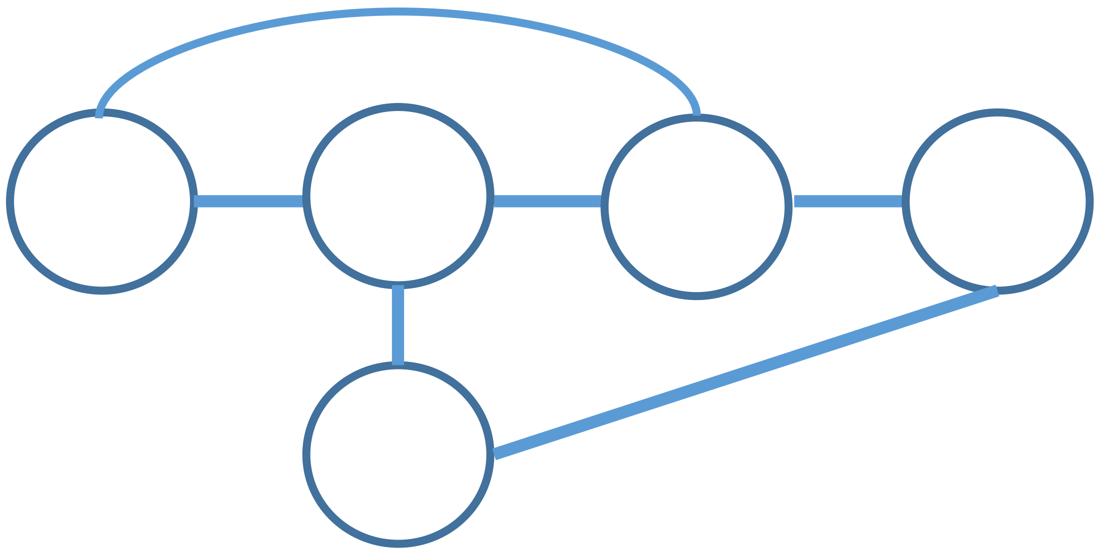
Vertex Cover

- Given a graph $G(V, E)$ and an integer k, does there exist a subset of V of size k which intersects all edges of E?



Reduction

- Arbitrarily label the vertices $1,2,\ldots,n$, and direct edges in topological ordering.



Reduction

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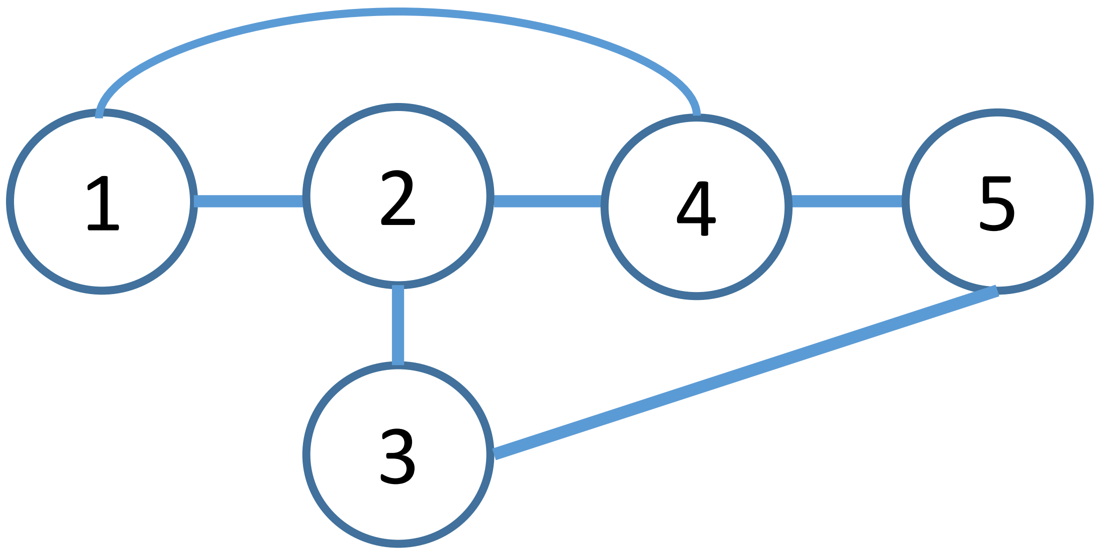
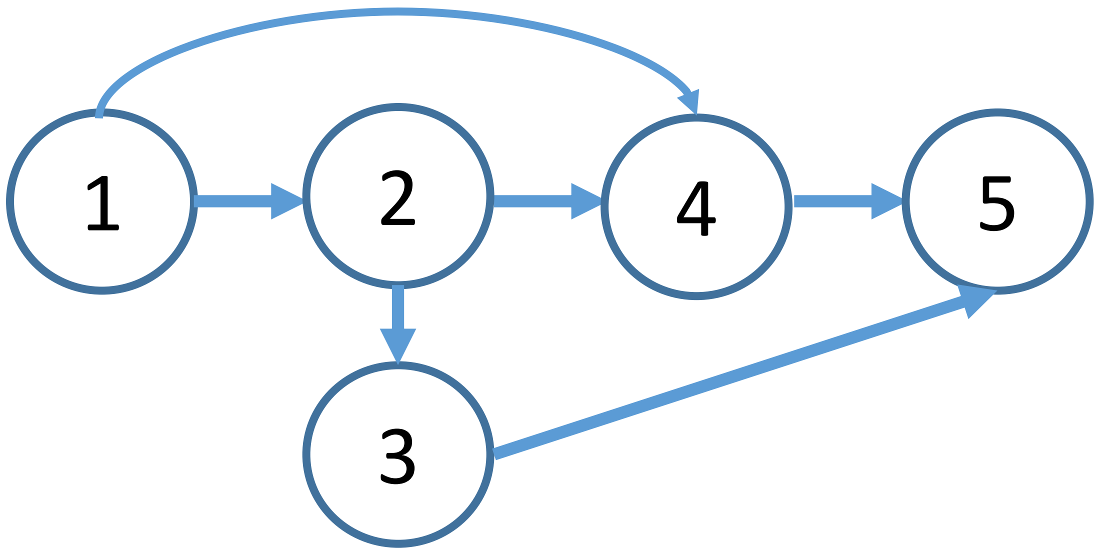


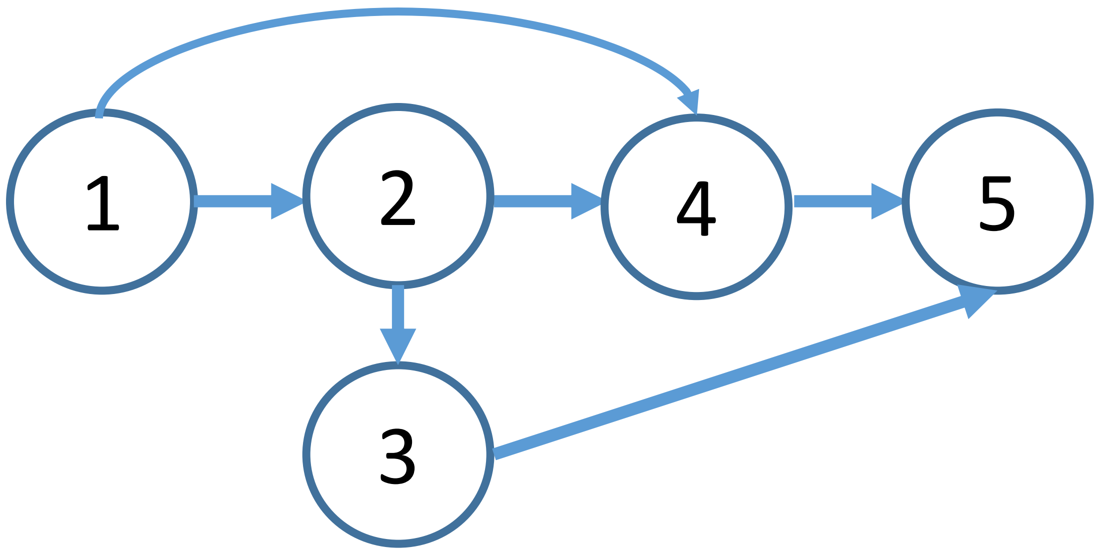
Reduction

- Arbitrarily label the vertices $1,2,\ldots,n$, and direct edges in topological ordering.

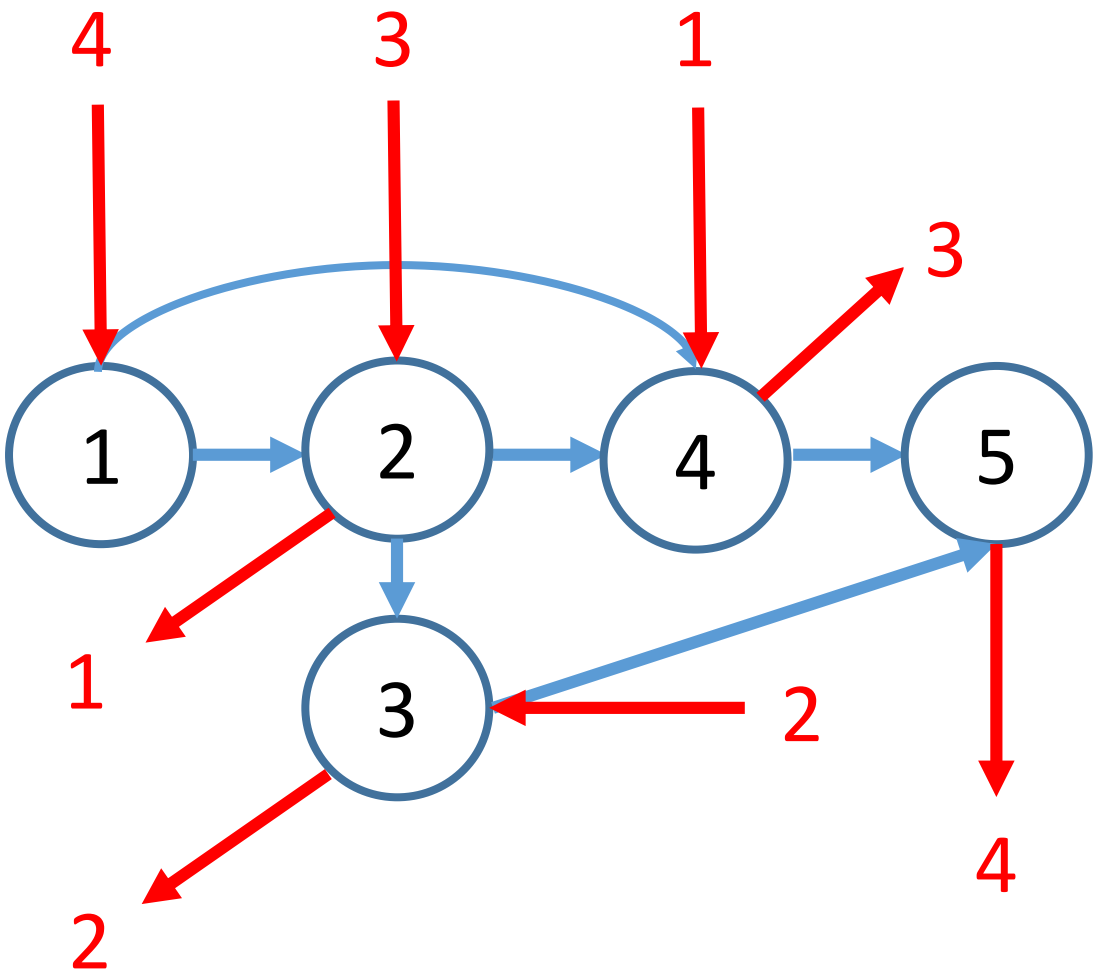


Reduction

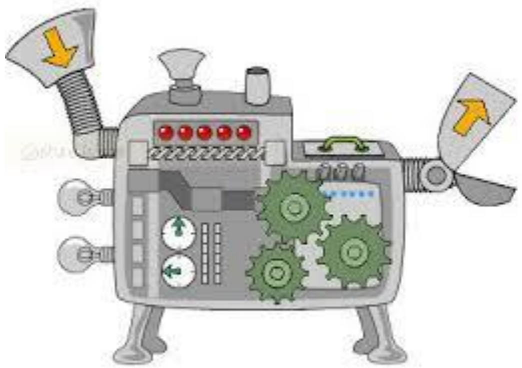
❖ For vertex i, create an ingoing path of length $i - 1$ and an outgoing path of length $n - i$.



Reduction



Path of length n if and only if a blue edge remains!



(1) Variables: For $1 \leq v \leq n$ and $0 \leq t \leq n^2$,

   (a) Integer Program: $x_v^t \in \{0, 1\}$

   (b) Relaxed Linear Program: $0 \leq x_v^t \leq 1$

(2) Goal (minimize cumulative pebbling cost): $\min \sum_{v \in V} \sum_{t=0}^{n^2} x_v^t$.

(3) Constraint 1 (Must Finish): $\sum_{t=0}^{n^2} x_n^t \geq 1$.

(4) Constraint 2 (No Pebbles At Start): $\sum_{v>0} x_v^0 \leq 0$.

(5) Constraint 3 (Pebbling Is Valid): For all v s.t $|\mathsf{Parents}(v)| \geq 1$ and $0 \leq t \leq n^2 - 1$ we have

$$x_v^{t+1} \leq x_v^t + \frac{\sum_{v' \in \mathsf{Parents}(v)} x_{v'}^t}{|\mathsf{Parents}(v)|} \ .$$

Fig. 5: Integer Program for Pebbling.

Theorem 9 *Let* G *be with constant indegree* δ*. Then there is a fractional solution to our LP Relaxation (of the Integer Program in Figure 5) with cost at most* $3n$*.*

Summary

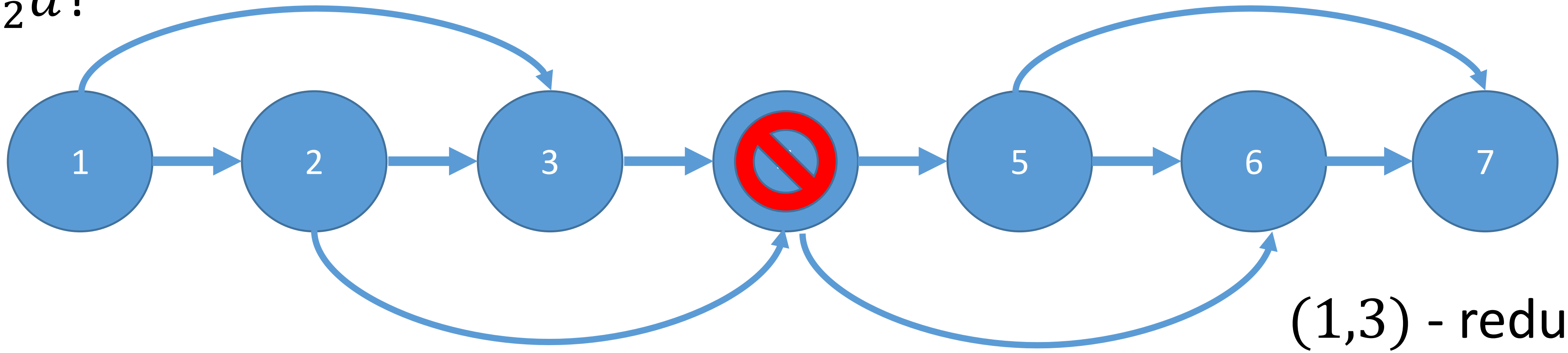
- We show computing $cc(G)$ is NP-hard (as is computing $st(G)$).
- Linear Program for $cc(G)$ has $\Omega\left(\frac{n}{\log n}\right)$ integrality gap.
- We show that given e, d, it is NP-hard to determine whether a graph is (e, d)-reducible (even for graphs with bounded degree).
- Given d, it may be NP-hard to:
  - Approximate e to a factor of 1.3 (minimum Vertex Cover).
  - Approximate e to a factor of 2 (Unique Games Conjecture).
- An optimal cumulative cost pebbling of a graph may take more than n steps.

Structure of Talk

- ✓ Background
- ✓ Graph Pebbling
- ✓ “Graph Reducibility”
- ❖ Open Problems

Open Questions

- Does there exist an algorithm to approximate $cc(G)$?
- Do there exist constants c_1, c_2 so that given an (e, d)-reducible graph, we can a set S of $c_1 e$ nodes such that $G - S$ has depth at most $c_2 d$?



$(1,3)$ - reducible

- Does there exist a graph with $cc(G) = \frac{n^2}{\log n}$ and space 3? (ADNV17)


thank you
danke
謝謝
ngiyabonga
teşekkür ederim
рахмат
Баярлалаа
спасибо
mersi
kia ora
barka
welalin
tack
spas
mahalo
tapadh leat
faafetai lava
vinaka
blagodaram
dank je
misaotra
matondo
paldies
grazzi
gracias
спасибі
kiitos
dankie
nanni
nandri
dhanyavad
mauruuru
köszönöm
ačiū
dankon
akun
хвала
asante
manana
obrigada
murakoze
tenki
bedankt
bayarlalaa
gràçie
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ありがとう
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obrigado
mèsi
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najis tuke
terima kasih
tanemirt
grazie
rahmet
kam sah hamnida
rahmat
ευχαριστώ
diolch
dhanyavadagalu
shukriya
mercé
мерси
তোমাকে ধন্যবাদ
감사합니다
xièxie
merci

Questions?